

On Bost-Connes type systems and Complex Multiplication

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Abstract

By using the theory of Complex Multiplication for general Siegel modular varieties we construct arithmetic subalgebras for BC-type systems attached to number fields containing a CM field. The abelian extensions obtained in this way are characterized by results of [Wei]. Our approach is based on a general construction of BC-type systems of Ha and Paugam [HP05] and extends the construction of the arithmetic subalgebra of Connes, Marcolli and Ramachandran [CMR05] for imaginary quadratic fields.

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1 Introduction

In their fundamental article [BC95] Bost and Connes constructed a quantum statistical mechanical system, the so called Bost-Connes or BC system, that recovers (among other arithmetic properties of \mathbb{Q}) the explicit class field theory of \mathbb{Q} . It is a natural question to ask for other quantum statistical mechanical systems that recover - at least partially - the (explicit) class field theory of number fields different from \mathbb{Q} .

This paper is a contribution to this question in the case of CM fields. In nature a CM field arises as (rational) endomorphism ring of an Abelian variety with Complex Multiplication.

Background

A quantum statistical mechanical system $\mathcal{A} = (A, (\sigma_t)_{t \in \mathbb{R}})$ is a C^* -algebra A together with a one-parameter group of automorphisms $(\sigma_t)_{t \in \mathbb{R}}$. Sometimes we call \mathcal{A} simply a C^* -dynamical system. One should think of A as the algebra of observables of a quantum physical system with the one-parameter group $(\sigma_t)_{t \in \mathbb{R}}$ implementing the time evolution of the observables. A symmetry of \mathcal{A} is a C^* -automorphism of A which commutes with the time evolution. Interesting properties of \mathcal{A} can be read off its equilibrium states at a temperature $T \in [0, \infty)$. An equilibrium state at inverse temperature $\beta = \frac{1}{T}$ is given by a so called KMS_β -state. See [BR79] and [BR81] for more information.

Now a first step towards generalizing the Bost-Connes system to other number fields is given by a system of the following kind (see [CMR06] and [LLN09])

A **BC-type system** for a number field K is a quantum statistical mechanical system $\mathcal{A} = (A, (\sigma_t)_{t \in \mathbb{R}})$ with the following properties

- (i) The partition function of \mathcal{A} is given by the Dedekind zeta function of K .
- (ii) The quotient of the idele class group C_K by the connected component D_K of the identity of C_K acts as symmetries on \mathcal{A} .
- (iii) For each inverse temperature $0 < \beta \leq 1$ there is a unique KMS_β -state.
- (iv) For each $\beta > 1$ the action of the symmetry group C_K/D_K on the set of extremal KMS_β -states is free and transitive.

Remark 1. i) Using class field theory we see that property (2) simply states that the Galois group $Gal(K^{ab}/K) \cong C_K/D_K$ of the maximal abelian extension K^{ab} of K is acting by symmetries on \mathcal{A} .

ii) Properties (3) and (4) say that there is a spontaneous symmetry breaking phenomenon at $\beta = 1$ (see [BC95] or pp. 400 in the book [CM08] of Connes and Marcolli).

Thanks to the work of Ha and Paugam [HP05] BC-type systems are known to exist for an arbitrary number field K , we denote their solution (see the next section for the definition) by

$$\mathcal{A}_K = (A_K, (\sigma_t)_{t \in \mathbb{R}}). \quad (1)$$

The paper of Laca, Larsen and Neshveyev [LLN09] gives a different description of \mathcal{A}_K . In fact [HP05] yields that \mathcal{A}_K fulfills the first two properties and [LLN09] verifies the last two properties.

We should mention that the approach of [HP05] works much more general, it allows to attach a quantum statistical mechanical system to an arbitrary Shimura variety; the above construction being a special case. The Shimura-theoretic approach of [HP05] is based on the foundational work of Connes, Marcolli and Ramachandran (see [CMR05] and [CMR06]).

The notion of an **arithmetic subalgebra** for a BC-type system, first encountered in [BC95], hints to a possible relation between seemingly unrelated areas, namely it asks for a connection between quantum statistical mechanical systems (physics/operator algebras) and class field theory (number theory). More precisely (see [CMR06])

A **BC-system** for a number field K is a BC-type system $\mathcal{A} = (A, (\sigma_t)_{t \in \mathbb{R}})$ such that

- (v) There is a K -rational subalgebra A^{arith} of A , called arithmetic subalgebra of \mathcal{A} , such that for every extremal KMS_∞ -state ϱ and every $f \in A^{arith}$ we have

$$\varrho(f) \in K^{ab} \quad (2)$$

and further K^{ab} is generated over K in this manner, i.e.

$$K^{ab} = K(\varrho(f) \mid \varrho \text{ extremal } KMS_\infty\text{-state, } f \in A^{arith}). \quad (3)$$

- (vi) Let $\nu \in C_K$ be a symmetry of \mathcal{A} , ϱ an extremal KMS_∞ -state of \mathcal{A} and $f \in A^{arith}$ and denote further by $[\nu] \in Gal(K^{ab}/K)$ the image of ν under Artin's reciprocity morphism $C_K \rightarrow Gal(K^{ab}/K)$. If we denote the action (given by pull-back) of ν on ϱ by ${}^\nu\varrho$, we have the following compatibility relation

$${}^\nu\varrho(f) = [\nu]^{-1}(\varrho(f)) \in K^{ab}. \quad (4)$$

Now the difficulty of constructing BC-systems comes from its relation with Hilbert's 12th problem, which asks for an explicit class field theory of a number field K . This problem is completely solved only in two cases, namely the case of $K = \mathbb{Q}$ and the case of K equal to an imaginary quadratic field, like $K = \mathbb{Q}(i)$. It does not come as a surprise that BC-systems are so far known to exist only

in these two cases. For $K = \mathbb{Q}$ see [BC95] and for the imaginary quadratic case we refer to [CMR05] and the very detailed exposition given in [CM08] pp. 551.

The construction in [CMR05] is based on the theory of Complex Multiplication (see section 4) which is a part of the arithmetic theory of Shimura Varieties. This theory allows to construct explicitly abelian extensions of higher dimensional generalizations of imaginary quadratic number fields, so called CM fields. A CM field E is a totally imaginary quadratic extension of a totally real number field. So for example cyclotomic number fields $\mathbb{Q}(\zeta_n)$, ζ_n being a primitive n -th root of unity, are seen to be CM-fields.

Abelian extensions of a CM field E are obtained by evaluating arithmetic Modular functions on so called CM-points on a Siegel upper half plane (see section 4.1 for more information).

Except for the case of an imaginary quadratic field, it is unfortunately not possible to generate the maximal abelian extension E^{ab} of a CM field E in this way, but still a non trivial abelian extension of infinite degree over E . We denote the latter abelian extension of E by

$$E^c \subset E^{ab}. \quad (5)$$

See Thm. 4.2 for the characterization of E^c given in [Wei]. As we remarked above for E equal to an imaginary quadratic field there is an equality $E^c = E^{ab}$.

Due to the lack of a general knowledge of the explicit class field theory of an arbitrary number field K we content ourself with the following weakening of a BC-system:

Let $F \subset K^{ab}$ be an arbitrary abelian extension of K . A **partial BC-system** for the extension F/K is defined like a BC system $\mathcal{A} = (A, (\sigma_t)_{t \in \mathbb{R}})$ for K except that we do not demand the arithmetic subalgebra A^{arith} of A to generate the maximal abelian extension K^{ab} of K but instead the abelian extension of F , i.e. in property (v) we replace K^{ab} by F and call the corresponding K -rational subalgebra A^{arith} a (partial) arithmetic subalgebra of \mathcal{A} .

Statement of our result

Now the aim of our paper is to prove the following

Theorem 1.1. *Let K be a number field containing a CM field. Denote by E the maximal CM field contained in K and define the abelian extension K^c of K to be the compositum*

$$K^c = K \cdot E^c \quad (6)$$

(with the notation from (5)).

Then the BC-type system \mathcal{A}_K of [HP05] (see (6.1)) is a partial BC-system for the extension K^c/K , i.e. there exists a (partial) arithmetic subalgebra A_K^{arith} for \mathcal{A}_K .

We want to explain our construction of A_K^{arith} (cf. section 7). It is inspired by the construction given in [CMR05] and [CMR06] (see also pp. 551 [CM08]).

Idea of our construction

Let (G, X, h) be a Shimura datum and denote by $Sh(G, X, h)$ the associated Shimura variety. In [HP05] the authors associate to the datum (G, X, h) , the variety $Sh(G, X, h)$ and some additional data a quotient map

$$U \longrightarrow Z \quad (7)$$

between a topological groupoid U and a quotient $Z = \Gamma \backslash U$ for a group Γ . Out of this data they construct a C^* -dynamical system $\mathcal{A} = (A, (\sigma_t)_{t \in \mathbb{R}})$. A dense $*$ -subalgebra H of A is thereby given by the compactly supported, continuous functions

$$H = C_c(Z) \quad (8)$$

on Z , where the groupoid structure of U induces the $*$ -algebra structure on H .

Moreover there are two variations of the above quotient map which can be put into the following (commutative) diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad} & Z \\ \uparrow & & \uparrow \\ U^+ & \xrightarrow{\quad} & Z^+ \\ \downarrow & & \downarrow \\ U^{ad} & \xrightarrow{\quad} & Z^{ad}. \end{array} \quad (9)$$

For more information see section 5. We will apply this general procedure in two special cases. For this we fix a number field K together with a maximal CM subfield E .

I) The BC-type system \mathcal{A}_K

The (0-dimensional) Shimura datum $\mathcal{S}_K = (T^K, X_K, h_K)$ gives rise to a quotient map denoted by

$$U_K \longrightarrow Z_K.$$

In this case the groupoid U_K is of the form (cf. (13))

$$U_K = T^K(\mathbb{A}_f) \boxplus (\hat{\mathcal{O}}_K \times Sh(T^K, X_K, h_K)).$$

The associated C^* -dynamical system is denoted by

$$\mathcal{A}_K = (A_K, (\sigma_t)_{t \in \mathbb{R}}). \quad (10)$$

It gives rise to a BC-type system for K . For the precise definition of \mathcal{A}_K and its properties (e.g. symmetries, extremal KMS_∞ -states) we refer the reader to section 3.1 and 6.1. Moreover we denote by H_K the dense subalgebra of A_K given by

$$H_K = C_c(Z_K). \quad (11)$$

II) The Shimura system \mathcal{A}_{Sh}

To the CM field E we associate the Shimura datum

$$\mathcal{S}_{Sh} = (GSp(V_E, \psi_E), \mathbb{H}_g^\pm, h_{cm})$$

where in fact the construction of the morphism h_{cm} takes some time (see 3.2). This is due to two difficulties that arise in the case of a general CM field E which are not visible in the case of imaginary quadratic fields. On the one hand one has to use the Serre group S^E and on the other hand in general the reflex field E^* of a CM field E is not anymore equal to E (see B.3). We denote the associated quotient map by

$$U_{Sh} \longrightarrow Z_{Sh}$$

and analogously its variations (see 6.2.2). Here the relevant groupoids in hand are of the following form

$$U_{Sh} = GSp(\mathbb{A}_f) \boxplus (\Gamma_{Sh,M} \times Sh(GSp(V_E, \psi_E), \mathbb{H}_g^\pm, h_{cm}))$$

and

$$U_{Sh}^{ad} = GSp^{ad}(\mathbb{Q})^+ \boxplus (\Gamma_{Sh,M}^{ad} \times \mathbb{H}_g).$$

We denote the resulting C^* -dynamical system by \mathcal{A}_{Sh} and call it Shimura system (cf. section 5.3).

Remark 2. In the case of an imaginary quadratic field K the Shimura system \mathcal{A}_{Sh} gives rise to the GL_2 -system of Connes and Marcolli [CM08].

The second system is of great importance for us because of the following: Denote by $x_{cm} \in \mathbb{H}_g$ the CM-point associated with h_{cm} and denote by \mathcal{M}^{cm} the ring of arithmetic Modular functions on \mathbb{H}_g defined at x_{cm} . By the theory of Complex Multiplication we know that for every $f \in \mathcal{M}^{cm}$ we have (cf. (6) and 4.3.1)

$$f(x_{cm}) \in K^c \subset K^{ab}$$

and moreover K^c is generated in this way. Our idea is now that \mathcal{M}^{cm} gives rise to the arithmetic subalgebra A_K^{arith} . More precisely we will construct a (commutative) diagram (see 6.3)

$$\begin{array}{ccc} U_K & \longrightarrow & Z_K \\ \downarrow & & \downarrow \\ U_{Sh} & \longrightarrow & Z_{Sh} \end{array} \quad (12)$$

which is induced by a morphism of Shimura data $\mathcal{S}_K \rightarrow \mathcal{S}_{Sh}$ constructed in section 3.3.

Then, using criterion 5.1, we see that the morphism $Z_{Sh}^+ \rightarrow Z_{Sh}$ (from the above diagram) is invertible and obtain in this way a continuous map

$$\Theta : Z_K \longrightarrow Z_{Sh} \longrightarrow Z_{Sh}^+ \longrightarrow Z_{Sh}^{ad}.$$

Using easy properties of the automorphism group of \mathcal{M}^{cm} , we can model each f in \mathcal{M}^{cm} as a function \tilde{f} on the space Z_{Sh}^{ad} (which might have singularities). Nevertheless in Prop. 7.1 we see that for every $f \in \mathcal{M}^{cm}$ the pull back $\tilde{f} \circ \Theta$ lies in $H_K = C_c(Z_K)$ and we can define A_K^{arith} as the K -algebra generated by these elements, i.e.

$$A_K^{arith} = \langle \tilde{f} \circ \Theta \mid f \in \mathcal{M}^{cm} \rangle_K.$$

Now, using the classification of extremal KMS_∞ -states of \mathcal{A}_K (see section 6.1.5), the verification of property (5') is an immediate consequence of our construction and property (6) follows by using Shimura's reciprocity law and the observation made in Prop. 4.4 (see section 7.1 for the details).

Our paper is organized along the lines of this section, recalling on the way the necessary background. In addition we put some effort in writing a long Appendix which covers hopefully enough information to make this paper "readable" for a person which has little beforehand knowledge of the arithmetic theory of Shimura varieties.

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Notations and conventions

We use the common notations \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} . If A denotes a ring or monoid, we denote its group of multiplicative units by A^\times . A number field is a finite extension of \mathbb{Q} . The ring of integers of a number field K is denoted by \mathcal{O}_K . We denote by $\mathbb{A}_K = \mathbb{A}_{K,f} \times \mathbb{A}_{K,\infty}$ the adele ring of K (with its usual topology), where $\mathbb{A}_{K,f}$ denotes the finite adeles and $\mathbb{A}_{K,\infty}$ the infinite adeles of K . \mathbb{A}_K contains K by the usual diagonal embedding and by $\widehat{\mathcal{O}}_K$ we denote the closure of \mathcal{O}_K in $\mathbb{A}_{K,f}$. Invertible adeles are called ideles. The idele class group $\mathbb{A}_K^\times / K^\times$ of K is denoted by C_K , its connected component of the identity by D_K . We fix an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} in \mathbb{C} . Usually we think of a number field K as lying in \mathbb{C} by an embedding $\tau : K \rightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$. Complex conjugation on \mathbb{C} is denoted by ι . Sometimes we write z^ι for the complex conjugate of a complex

number z .

Artin's reciprocity map $\mathbb{A}_K^\times \rightarrow \text{Gal}(K^{ab}/K) : \nu \mapsto [\nu]$ is normalized such that an uniformizing parameter maps to the arithmetic Frobenius element. Further given a group G acting partially on a set X we denote by

$$G \boxtimes X = \{(g, x) \in G \times X \mid gx \in X\} \quad (13)$$

the corresponding groupoid (see p. 327 [LLN09]).

If X denotes a topological space we write $\pi_0(X)$ for its set of connected components.

2 On the arithmetic subalgebra for $K = \mathbb{Q}(i)$

Before we describe our general construction we will explain the easiest case $K = \mathbb{Q}(i)$, where many simplifications occur, in some detail and point out the modifications necessary for the general case. For the remainder of this section K always denotes $\mathbb{Q}(i)$, although many of the definitions work in general. For the convenience of the reader we will try to make the following section as self-contained as possible.

2.1 The quotient map $U_K \rightarrow Z_K$

We denote by T^K the \mathbb{Q} -algebraic torus given by the Weil restriction $T^K = \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$ of the multiplicative group $\mathbb{G}_{m,K}$, i.e. for a \mathbb{Q} -algebra R the R -points of T^K are given by $T^K(R) = (R \otimes_{\mathbb{Q}} K)^\times$. In particular we see that $T^K(\mathbb{Q}) = K^\times$, $T^K(\mathbb{A}_f) = \mathbb{A}_{K,f}^\times$ and $T^K(\mathbb{R}) = \mathbb{A}_{K,\infty}^\times$. In our special case we obtain that after extending scalars to \mathbb{R} the \mathbb{R} -algebraic group $T_{\mathbb{R}}^K$ is isomorphic to $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{R}})$. Further the finite set $X_K = T^K(\mathbb{R})/T^K(\mathbb{R})^+ = \pi_0(T^K(\mathbb{R}))$ consists in our case of only one point. With this in mind we consider the 0-dimensional Shimura datum (see D.5)

$$\mathcal{S}_K = (T^K, X_K, h_K) \quad (14)$$

where the morphism $h_K : \mathbb{S} \rightarrow T_{\mathbb{R}}^K$ is simply given by the identity (thanks to $T_{\mathbb{R}}^K \cong \mathbb{S}$). (In the general case h_K is chosen accordingly to Lemma 3.1.)

The (0-dimensional) Shimura variety $Sh(\mathcal{S}_K)$ is in our case of the simple form

$$Sh(\mathcal{S}_K) = T^K(\mathbb{Q}) \backslash (X_K \times T^K(\mathbb{A}_f)) = K^\times \backslash \mathbb{A}_{K,f}^\times. \quad (15)$$

We write $[z, l]$ for an element in $Sh(\mathcal{S}_K)$ meaning that $z \in X_K$ and $l \in T^K(\mathbb{A}_f)$. (For general number fields the description of $Sh(\mathcal{S}_K)$ is less explicit but no difficulty occurs.)

Remark 3. The reader should notice that by class field theory we can identify $Sh(\mathcal{S}_K) = K^\times \backslash \mathbb{A}_{K,f}^\times$ with the Galois group $\text{Gal}(K^{ab}/K)$ of the maximal abelian extension K^{ab} of K . This is true in general, see section 6.1.4.

The (topological) groupoid U_K underlying the BC-type system $\mathcal{A}_K = (A_K, (\sigma_t)_{t \in \mathbb{R}})$ is now of the form (see (13) for the notation)

$$U_K = T^K(\mathbb{A}_f) \boxplus (\widehat{\mathcal{O}}_K \times Sh(\mathcal{S}_K)) \quad (16)$$

with the natural action of $T^K(\mathbb{A}_f)$ on $Sh(\mathcal{S}_K)$ (see D.2) and the partial action of $T^K(\mathbb{A}_f) = \mathbb{A}_{K,f}^\times$ on the multiplicative semigroup $\widehat{\mathcal{O}}_K \subset \mathbb{A}_{K,f}$ by multiplication. The group

$$\Gamma_K^2 = \widehat{\mathcal{O}}_K^\times \times \widehat{\mathcal{O}}_K^\times \quad (17)$$

is acting on U_K as follows

$$(\gamma_1, \gamma_2)(g, \rho, [z, l]) = (\gamma_1^{-1} g \gamma_2, \gamma_2 \rho, [z, l \gamma_2^{-1}]), \quad (18)$$

where $\gamma_1, \gamma_2 \in \widehat{\mathcal{O}}_K^\times$, $g, l \in T^K(\mathbb{A}_f)$, $\rho \in \widehat{\mathcal{O}}_K$ and $z \in X_K$ and we obtain the quotient map

$$U_K \longrightarrow Z_K = \Gamma_K^2 \backslash U_K. \quad (19)$$

In the end of this section we will construct the arithmetic subalgebra A_K^{arith} of the BC-type system \mathcal{A}_K which is contained in $H_K = C_c(Z_K) \subset A_K$. For this we will need

2.2 The quotient map $U_{Sh} \longrightarrow Z_{Sh}$

In our case of $K = \mathbb{Q}(i)$ the maximal CM subfield E of K is equal to K . The Shimura datum \mathcal{S}_{Sh} associated with E is of the form (see 3.2)

$$\mathcal{S}_{Sh} = (GSp(V_E, \psi_E), \mathbb{H}^\pm, h_{cm}). \quad (20)$$

Here $GSp = GSp(V_E, \psi_E)$ is the general symplectic group (cf. D.3) associated with the symplectic vector space (V_E, ψ_E) .

The latter is in general chosen accordingly to (55). Due to the fact that the reflex field E^* (cf. B.3) is equal to E and the Serre group S^E is equal to $T^E = T^K$ we can simply choose the \mathbb{Q} -vector space V_E to be the \mathbb{Q} -vector space E and the symplectic form $\psi_E : E \times E \rightarrow \mathbb{Q}$ to be the map $(x, y) \mapsto Tr_{E/\mathbb{Q}}(ixy^t)$. A simple calculation shows that $\psi_E(f(x), f(y)) = \det(f)\psi_E(x, y)$, for all $f \in End_{\mathbb{Q}}(V_E)$ and all $x, y \in V_E$, therefore we can identify GSp with $GL_2 = GL(V_E)$. Now again using the fact that the Serre group S^E equals T^E we see that the general construction of $h_{cm} : \mathbb{S} = T_{\mathbb{R}}^E \rightarrow GSp_{\mathbb{R}} = GL_{2, \mathbb{R}}$ (see (59)) is given on the \mathbb{R} -points by $a + ib \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in GL_2(\mathbb{R})$.

Each $\alpha \in GSp(\mathbb{R})$ defines a map $\alpha^{-1}h_{cm}\alpha : \mathbb{S} \rightarrow GSp_{\mathbb{R}}$ given on the \mathbb{R} -points by $a + ib \in \mathbb{C}^\times \mapsto \alpha^{-1}h_{cm}(a + ib)\alpha \in GL_2(\mathbb{R})$ and the $GSp(\mathbb{R})$ -conjugacy class $X = \{\alpha^{-1}h_{cm}\alpha \mid \alpha \in GSp_{\mathbb{R}}(\mathbb{R})\}$ of h_{cm} can be identified with the Siegel upper lower half space $\mathbb{H}^\pm = \mathbb{C} - \mathbb{R}$ by the map

$$\alpha^{-1}h_{cm}\alpha \in X \mapsto (\alpha^{-1}h_{cm}(i)\alpha) \cdot i \in \mathbb{H}^\pm,$$

where the latter action \cdot denotes Moebius transformation. Under this identification the morphism h_{cm} corresponds to the point $x_{cm} = i$ on the upper half plane \mathbb{H} . The point $x_{cm} \in \mathbb{H}$ is a so called CM-point (see D.6).

Remark 4. The definition of a CM-point and the observation made in (58) explain the need of using the Serre group in the general construction of h_{cm} . The explanation given in 4.1 shows in particular why we have to define the vector space V_E in general accordingly to (55).

The Shimura variety $Sh(\mathcal{S}_h)$ is of the nice form (cf. (136))

$$Sh(\mathcal{S}_h) = GSp(\mathbb{Q}) \backslash (\mathbb{H}^\pm \times GSp(\mathbb{A}_f)).$$

Again we write elements as $[z, l] \in Sh(\mathcal{S}_h)$ with $z \in \mathbb{H}^\pm$ and $g \in GSp(\mathbb{A}_f)$. In our case the topological groupoid U_{Sh} underlying the Shimura system \mathcal{A}_h is given by (cf. 6.2.1)

$$U_{Sh} = GSp(\mathbb{A}_f) \boxplus (M_2(\widehat{\mathbb{Z}}) \times Sh(\mathcal{S}_h)), \quad (21)$$

where $GSp(\mathbb{A}_f) = GL_2(\mathbb{A}_f)$ is acting in the natural way on $Sh(\mathcal{S}_h)$ and partially on the multiplicative monoid of 2×2 -matrices $M_2(\widehat{\mathbb{Z}}) \subset M_2(\mathbb{A}_f)$ with entries in $\widehat{\mathbb{Z}}$. The group

$$\Gamma_{Sh}^2 = GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}}) \quad (22)$$

is acting on U_{Sh} exactly like in (18) and induces the quotient map

$$U_{Sh} \longrightarrow Z_{Sh} = \Gamma_{Sh}^2 \backslash U_{Sh}. \quad (23)$$

Remark 5. Note that the quotient Z_{Sh} is not a groupoid anymore (see top of p. 251 [HP05]).

In our example it is sufficient to consider the positive groupoid U_{Sh}^+ (see 6.2.2) associated with U_{Sh} . (In the general case the adjoint groupoid U_{Sh}^{ad} seems to be more appropriate.) It is given by

$$U_{Sh}^+ = GSp(\mathbb{Q})^+ \boxplus (M_2(\widehat{\mathbb{Z}}) \times \mathbb{H}) \quad (24)$$

together with the group

$$(\Gamma_{Sh}^+)^2 = GL_2(\mathbb{Z})^+ \times GL_2(\mathbb{Z})^+ = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \quad (25)$$

acting by

$$(\gamma_1, \gamma_2)(g, \rho, z) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z) \quad (26)$$

and inducing the quotient map

$$U_{Sh}^+ \longrightarrow Z_{Sh}^+ = (\Gamma_{Sh}^+)^2 \backslash U_{Sh}^+. \quad (27)$$

By construction $GSp(\mathbb{R})$ is acting (free and transitively) on \mathbb{H}^\pm and $GSp(\mathbb{R})^+$, the connected component of the identity, can be thought of as stabilizer of the upper half plane $\mathbb{H}^+ = \mathbb{H}$ which explains the action of $GSp(\mathbb{Q})^+ = GSp(\mathbb{Q}) \cap GSp(\mathbb{R})^+$ on \mathbb{H} . Thanks to criterion 5.1 we know that the natural (equivariant) morphism of topological groupoids $U_{Sh}^+ \rightarrow U_{Sh}$ given by $(g, \rho, z) \mapsto (g, \rho, [z, 1])$ induces a homeomorphism on the quotient spaces $Z_{Sh}^+ \rightarrow Z_{Sh}$ in the commutative diagram

$$\begin{array}{ccc} U_{Sh} & \longrightarrow & Z_{Sh} \\ \uparrow & & \uparrow \cong \\ U_{Sh}^+ & \longrightarrow & Z_{Sh}^+ \end{array} \quad (28)$$

Remark 6. The groupoid U_{Sh}^+ corresponds to the GL_2 -system of Connes and Marcolli (see [CM08] and 5.8 [HP05])

2.3 A map relating U_K and U_{Sh}

We want to define an equivariant morphism of topological groupoids $U_K \rightarrow U_{Sh}$, where equivariance is meant with respect to the actions of Γ_K on U_K and Γ_{Sh} on U_{Sh} . For this it is necessary (and more or less sufficient) to construct a morphism of Shimura data between $\mathcal{S}_K = (T^K, X_K, h_K)$ and $\mathcal{S}_{Sh} = (GSp(V_E, \psi_E), \mathbb{H}^\pm, h_{cm})$, which is given by a morphism of algebraic groups

$$\varphi : T^K \rightarrow GSp \quad (29)$$

such that $h_{cm} = \varphi_{\mathbb{R}} \circ h_K$. In our case the general construction of φ , stated in (60), reduces to the simple map (on the \mathbb{Q} -points)

$$a + ib \in K^\times = T^K(\mathbb{Q}) \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in GL_2(\mathbb{Q}) = GSp(\mathbb{Q})$$

and we see in fact that after extending scalars to \mathbb{R} the morphism $\varphi_{\mathbb{R}} : T_{\mathbb{R}}^K = \mathbb{S} \rightarrow GSp_{\mathbb{R}}$ is already equal to $h_{cm} : \mathbb{S} \rightarrow GSp_{\mathbb{R}}$. The simplicity of our example comes again from the fact that we don't have to bother about the Serre group, which makes things less explicit, although the map $\varphi : T^K \rightarrow GSp$ still has a quite explicit description even in the general case thanks to Lemma 3.3.

Now by functoriality (see section D.5) we obtain a morphism of Shimura varieties

$$Sh(\varphi) : Sh(\mathcal{S}_K) \rightarrow Sh(\mathcal{S}_{Sh})$$

which can be explicitly described by

$$[z, l] \in K^\times \backslash (X_K \times T^K(\mathbb{A}_f)) \mapsto [x_{cm}, \varphi(\mathbb{A}_f)(l)] \in GSp(\mathbb{Q}) \backslash (\mathbb{H}^\pm \times GSp(\mathbb{A}_f)).$$

In the general case we have essentially the same description (see (105)), the point being that every element z in X_K is mapped to $x_{cm} \in \mathbb{H}^\pm$, as in the

general case. Using $\widehat{\mathcal{O}}_K = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_K$ we can continue the map $\varphi(\mathbb{A}_f)$ to a morphism of (topological) semigroups $M(\varphi)(\mathbb{A}_f) : \widehat{\mathcal{O}}_K \rightarrow M_2(\widehat{\mathbb{Z}})$ by setting $n \otimes (a + ib) \mapsto \begin{pmatrix} an & -bn \\ bn & an \end{pmatrix}$. By continuation we mean that $\varphi(\mathbb{A}_f)$ and $M(\varphi)(\mathbb{A}_f)$ agree on the intersection of $T^K(\mathbb{A}_f) \cap \widehat{\mathcal{O}}_K \subset \mathbb{A}_{K,f}$. In the general case the explicit description of φ given in (62) is used to continue φ to $M(\varphi)$ (see 6.3.1), the above example being a special case. Now it can easily be checked that

$$(g, \rho, [z, l]) \in U_K \mapsto (\varphi(\mathbb{A}_f)(g), M(\varphi)(\mathbb{A}_f)(\rho), [x_{cm}, \varphi(\mathbb{A}_f)(l)]) \in U_{Sh} \quad (30)$$

defines the desired equivariant morphism of topological groupoids

$$U_K \longrightarrow U_{Sh}. \quad (31)$$

Summarizing we obtain the following commutative diagram (using (28))

$$\begin{array}{ccc} U_K & \longrightarrow & Z_K \\ \downarrow & & \downarrow \\ U_{Sh} & \longrightarrow & Z_{Sh} \\ \uparrow & & \uparrow \cong \\ U_{Sh}^+ & \longrightarrow & Z_{Sh}^+ \end{array} \quad (32)$$

which gives us the desired morphism of topological spaces

$$\Theta : Z_K \longrightarrow Z_{Sh} \longrightarrow Z_{Sh}^+. \quad (33)$$

In general we have to go one step further and use the adjoint groupoid Z_{Sh}^{ad} (cf. 6.2.2) but this only due to the general description of the automorphism group $Aut_{\mathbb{Q}}(\mathcal{M})$ of the field of arithmetic automorphic functions \mathcal{M} , see 4.3.1 and the next section for explanations.

2.4 Interlude: Theory of Complex Multiplication

In this section we will provide the number theoretic background which is necessary to understand the constructions done so far.

We are interested in constructing the maximal abelian extension E^{ab} of $E = K = \mathbb{Q}(i)$ and there are in general two known approaches to this problem.

I) The elliptic curve $A : y^2 = x^3 + x$

Let us denote by A the elliptic curve defined by the equation

$$A : y^2 = x^3 + x. \quad (34)$$

It is known that the field of definition of A and its torsion points generate the maximal abelian extension E^{ab} of E . (By field of definition of the torsion points of A we mean the coordinates of the torsion points.)

Notice that in our case the complex points of our elliptic curve A are given by $A(\mathbb{C}) = \mathbb{C}/\mathcal{O}_E$ and the rational ring of endomorphisms of A turns out to be $\text{End}(A(\mathbb{C}))_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_E = E$, we say that A has complex multiplication by E . Remember that $\mathcal{O}_E = \mathbb{Z}[i]$.

Remark 7. To obtain the abelian extensions of K provided by A explicitly one may use for example the Weierstrass p -function associated with A (see [Sil94]) but as we will use another approach we don't want to dive into this beautiful part of explicit class field theory.

II) The (Siegel) Modular curve $Sh(GSp, \mathbb{H}^{\pm}, h_{cm})$

We want to interpret the Shimura variety $Sh(\mathcal{S}_h) = Sh(GSp, \mathbb{H}^{\pm}, h_{cm})$ constructed in the last section as moduli space of elliptic curves with torsion data.

The moduli theoretic picture

For this we consider the connected component $Sh^o = Sh(\mathcal{S}_h)^o$ of our Shimura variety which is described by the projective system (see D.4 and [Mil04])

$$Sh^o = \varprojlim_N \Gamma(N) \backslash \mathbb{H}, \quad (35)$$

where $\Gamma(N)$, for $N \geq 1$, denotes the subgroup of $\Gamma = \Gamma(1) = SL_2(\mathbb{Z})$ defined by $\Gamma(N) = \{g \in \Gamma \mid g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\}$. We can view the quotient $\Gamma(N) \backslash \mathbb{H}$ as a complex analytic space, but thanks to the work of Baily and Borel, it carries also a unique structure of an algebraic variety over \mathbb{C} (see [Mil04]). We will use both viewpoints. Seen as an analytic space we write $\mathbb{H}(N) = \Gamma(N) \backslash \mathbb{H}$ and for the algebraic space we write $Sh_N^o = \Gamma(N) \backslash \mathbb{H}$.

Observe now that the space $\mathbb{H}(N)$ classifies isomorphism classes of pairs (A, t) given by an elliptic curve A over \mathbb{C} together with a N -torsion point t of A . In particular $\mathbb{H}(1) = \Gamma \backslash \mathbb{H}$ classifies isomorphism classes of elliptic curves over \mathbb{C} .

In this picture our CM-point $[x_{cm}]_1 = [i]_1 \in \mathbb{H}(1)$ corresponds to the isomorphism class of the elliptic curve A from (34) and more general the points $[x_{cm}]_N \in \mathbb{H}(N)$ capture the field of definition of A and its various torsion points and recover the maximal abelian extension E^{ab} of E in this way! By $[z]_N$ we denote the image of $z \in \mathbb{H}$ in $\mathbb{H}(N)$ under the natural quotient map $\mathbb{H} \rightarrow \mathbb{H}(N)$.

Remark 8. For the relation between Sh and Sh^o we refer the reader to pp. 51 [Mil04].

The field of arithmetic Modular functions \mathcal{M}

To construct the abelian extensions provided by the various points $[x_{cm}]_N$ explicitly we proceed as follows: We consider the connected canonical model M^o of Sh^o (see D.7) which provides us with an algebraic model $M_N^o = \Gamma(N) \backslash M^o$ of the algebraic variety Sh_N^o over the cyclotomic field $\mathbb{Q}(\zeta_N)$. In general we obtain algebraic models over subfields of \mathbb{Q}^{ab} . This means M_N^o is an algebraic variety defined over the cyclotomic field $\mathbb{Q}(\zeta_N)$ and after scalar extension to \mathbb{C} it becomes isomorphic to the complex algebraic variety Sh_N^o . Let us denote by $k(M_N^o)$ the field of rational functions on M_N^o , in particular this means elements in $k(M_N^o)$ are rational over $\mathbb{Q}(\zeta_N)$. It makes sense to view the point $[x_{cm}]_N$ as a point on M_N^o and if a function $f \in k(M_N^o)$ is defined at $[x_{cm}]_N$, then we know (cf. 4.3.1)

$$f([x_{cm}]_N) \in E^{ab}. \quad (36)$$

In particular varying over the various N and the rational functions in $k(M_N^o)$ the values $f([x_{cm}]_N)$ generate E^{ab} over E . The next step is to realize that the function field $k(M_N^o)$ can be seen as subset of the field of rational functions $k(Sh_N^o)$ on the complex algebraic variety Sh_N^o (cf. 4.3.1). As rational functions in $k(Sh_N^o)$ correspond to meromorphic functions on $\mathbb{H}(N)$ and meromorphic functions on $\mathbb{H}(N)$ are nothing else than meromorphic functions on \mathbb{H} that are invariant under the action of $\Gamma(N)$ we can view each rational function in $k(M_N^o)$ as a meromorphic function on \mathbb{H} which is invariant under $\Gamma(N)$. If we denote by $k(M_N^o)_{cusp}$ the subfield of $k(M_N^o)$ consisting of functions $f \in k(M_N^o)$ that give rise to meromorphic functions on \mathbb{H} that are meromorphic at the cusps (see 4.3.1), it makes sense to define the field of meromorphic functions \mathcal{M} on \mathbb{H} given by the union

$$\mathcal{M} = \bigcup_N k(M_N^o)_{cusp}. \quad (37)$$

Due to (36) we know furthermore that for every $f \in \mathcal{M}$, which is defined in x_{cm} , we have

$$f(x_{cm}) \in K^{ab} = E^{ab} \quad (38)$$

and K^{ab} is generated in this way. Therefore we call the field \mathcal{M} the **field of arithmetic Modular functions**.

Explicitly \mathcal{M} is described for example in [Shi00] or [CM08] (Def. 3.60). A very famous arithmetic Modular function is given by the j -function which generates the Hilbert class field of an arbitrary imaginary quadratic field.

Remark 9. 1) In light of 4.1 and 3.2 2) we mention that the field of definition $E(x_{cm})$ of x_{cm} is in our example equal to $E = \mathbb{Q}(i)$, which is the reason why our example is especially simple.

2) If we take a generic meromorphic function g on $\mathbb{H}(N)$ that is defined in $[x_{cm}]_N$ then the value $g([x_{cm}]_N) \in \mathbb{C}$ will not even be algebraic. This is the reason why we need the canonical model M^o which provides an arithmetic structure for the field of meromorphic functions on $\mathbb{H}(N)$.

In the general case the construction of \mathcal{M} is quite similar to the construction above (see section 4), the only main difference being that in general the theory is much less explicit (e.g. the description of \mathcal{M}).

Automorphisms of \mathcal{M} and Shimura's reciprocity law

In our example $K = E = \mathbb{Q}(i)$, using the notation from 4.3, we have the equality $\overline{\mathcal{E}} = \frac{GSp(\mathbb{A}_f)}{\mathbb{Q}^\times}$ and obtain a group homomorphism

$$GSp(\mathbb{A}_f) \longrightarrow \overline{\mathcal{E}} \longrightarrow Aut_{\mathbb{Q}}(\mathcal{M}), \quad (39)$$

where the first arrow is simply the projection and the second arrow comes from 4.3.2. We denote the action of $\alpha \in GSp(\mathbb{A}_f)$ on a function $f \in \mathcal{M}$ by ${}^\alpha f$. In particular we see that $\alpha \in GSp(\mathbb{Q})^+ = SL_2(\mathbb{Q})$ is acting by

$${}^\alpha f = f \circ \alpha^{-1}, \quad (40)$$

where α^{-1} acts on \mathbb{H} by Mobius transformation. The adjoint system, which is well suited for the general case (cf. 6.2.2), is not needed in our special example. We note that the group $GSp^{ad}(\mathbb{Q})^+$ occurring in (6.2.2) is given by $SL_2(\mathbb{Q})/\{\pm I\}$, where I denotes the unit in $SL_2(\mathbb{Q})$. Due to the fact that $\{\pm I\}$ acts trivially on \mathbb{H} we can lift the action to $GSp(\mathbb{Q})^+$. Further there is a morphism of algebraic groups

$$\eta : T^K \rightarrow GSp \quad (41)$$

which induces a group homomorphism denoted by (compare (75))

$$\overline{\eta} = \eta(\mathbb{A}_f) : T^K(\mathbb{A}_f) \rightarrow GSp(\mathbb{A}_f). \quad (42)$$

The **reciprocity law of Shimura** can be stated in our special case as follows: Let ν be in $\mathbb{A}_{K,f}^\times$, denote by $[\nu] \in Gal(K^{ab}/K)$ its image under Artin's reciprocity map and let $f \in \mathcal{M}$ be defined in $x_{cm} \in \mathbb{H}$. Then $\overline{\eta}(\nu)f$ is also defined in $x_{cm} \in \mathbb{H}$ and

$$\overline{\eta}(\nu)f(x_{cm}) = [\nu]^{-1}f(x_{cm}) \in K^{ab}. \quad (43)$$

The formulation in the general case concentrates on the group $\overline{\mathcal{E}}$ (see 4.4).

Now we are ready for the

2.5 Construction of the arithmetic subalgebra

Define \mathcal{M}^{cm} to be the subring of functions in \mathcal{M} which are defined at $x_{cm} \in \mathbb{H}$. For every $f \in \mathcal{M}^{cm}$ we define a function \tilde{f} on the groupoid U_{Sh}^+ by

$$\tilde{f}(g, \rho, z) = \begin{cases} {}^\rho f(z) & \rho \in GSp(\widehat{\mathbb{Z}}) \\ 0 & \rho \in M_2(\widehat{\mathbb{Z}}) - GSp(\widehat{\mathbb{Z}}) \end{cases} \quad (44)$$

Thanks to (40) \tilde{f} is invariant under the action of $\Gamma_{Sh}^+ = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ and therefore \tilde{f} descends to the quotient Z_{Sh}^+ (cf. (27)). Proposition 7.1 shows further that the pull-back $\tilde{f} \circ \Theta$ (see (33)) defines a compactly supported, continuous function on Z_K , i.e. we have $\tilde{f} \circ \Theta \in H_K = C_c(Z_K) \subset A_K$. Therefore we can define the K -subalgebra A_K^{arith} of H_K generated by these elements

$$A_K^{arith} = \langle \tilde{f} \circ \Theta \mid f \in \mathcal{M}^{cm} \rangle_K. \quad (45)$$

Now we want to show that A_K^{arith} is indeed an arithmetic subalgebra for \mathcal{A}_K (see (2)).

The set \mathcal{E}_∞ of extremal KMS_∞ -states of \mathcal{A}_K is indexed by the set $Sh(\mathcal{S}_K)$ and for every $\omega \in Sh(\mathcal{S}_K)$ the corresponding KMS_∞ -state ϱ_ω is given on an element $f \in H_K = C_c(Z_K)$ by (cf. 6.1.5)

$$\varrho_\omega(f) = f(1, 1, \omega). \quad (46)$$

Using remark 3 we write $[\omega]$ for an element $\omega \in Sh(\mathcal{S}_K)$ when regarded as element in $Gal(K^{ab}/K)$. Now if we take a function $f \in \mathcal{M}^{cm}$ and $\omega \in Sh(\mathcal{S}_K)$ we immediately see

$$\varrho_\omega(\tilde{f} \circ \Theta) = [\omega]^{-1}(f(x_{cm})) \in K^{ab} \quad (47)$$

and property (v) follows (in general we will only show property (v') of course). To show property (vi) we take a symmetry $\nu \in C_K = \mathbb{A}_K^\times / K^\times$ (see 6.1.4) of \mathcal{A}_K and denote by $[\nu] \in Gal(K^{ab}/K)$ its image under Artin's reciprocity homomorphism and let f and ω be as above. We denote the action (pull-back) of ν on ϱ_ω by ${}^\nu\varrho_\omega$ and obtain

$${}^\nu\varrho_\omega(\tilde{f} \circ \Theta) = \varphi(\mathbb{A}_f)(\nu) f(x_{cm}). \quad (48)$$

But thanks to our observation Proposition 4.4 we know that

$$\varphi(\mathbb{A}_f)(\nu) = \overline{\eta}(\nu) \in GSp(\mathbb{A}_f) \quad (49)$$

and now thanks to Shimura's (43) we can conclude

$${}^\nu\varrho_\omega(\tilde{f} \circ \Theta) = \overline{\eta}(\nu) f(x_{cm}) = [\nu]^{-1}(f(x_{cm})) = [\nu]^{-1}(\varrho_\omega(\tilde{f} \circ \Theta)) \in K^{ab} \quad (50)$$

which proves property (vi). For the general case and more details we refer the reader to 7.1.

Remark 10. Our arithmetic subalgebra \mathcal{A}_K in the case of $K = \mathbb{Q}(i)$ is essentially the same as in [CMR05].

Remark 11. In a fancy (and very sketchy) way we might say that the two different pictures, one concentrating on the single elliptic curve A the other on the moduli space of elliptic curves (see the beginning of section 2.4), are related via the Langlands correspondence. In terms of Langlands correspondence, the single elliptic curve A lives on the motivic side whereas the moduli space of

elliptic curves lives (partly) on the automorphic side. As we used the second picture for our construction of an arithmetic subalgebra, we might say that our construction is automorphic in nature. This explains the fact that we have a "natural" action of the idele class group on our arithmetic subalgebra (see above). Using the recent theory of endomotives (see chapter 4 and in particular p. 551 [CM08]) one can recover the arithmetic subalgebra A_K^{arith} by only using the single elliptic curve A . In particular one obtains a natural action of the Galois group. This and more will be elaborated in another paper.

We now concentrate on the general case.

3 Two Shimura data and a map

As throughout the paper let K denote a number field and E its maximal CM subfield. We fix an embedding $\tau : K \rightarrow \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ and denote complex conjugation on \mathbb{C} by ι .

To K , resp. E , we will attach a Shimura datum \mathcal{S}_K , resp. \mathcal{S}_h , and show how to construct a morphism $\varphi : \mathcal{S}_K \rightarrow \mathcal{S}_h$ between them. We will freely use the Appendix: every object not defined in the following can be found there or in the references given therein.

Recall that the Serre group attached to K is denoted by S^K (cf. C.1), it is a quotient of the algebraic torus T^K (defined below), the corresponding quotient map is denoted by $\pi^K : T^K \rightarrow S^K$.

3.1 Protagonist I: \mathcal{S}_K

The 0-dimensional Shimura datum $\mathcal{S}_K = (T^K, X_K, h_K)$ (see section D.5) is given by the Weil restriction $T^K = \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$, the discrete space $X_K = T^K(\mathbb{R})/T^K(\mathbb{R})^+$ and a morphism $h_K : \mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \rightarrow T_{\mathbb{R}}^K$ which is chosen accordingly to the next

Lemma 3.1. *There is a morphism of algebraic groups $h_K : \mathbb{S} \rightarrow T_{\mathbb{R}}^K$ such that the diagram*

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_K} & T_{\mathbb{R}}^K \\ & \searrow h^K & \downarrow \pi_{\mathbb{R}}^K \\ & & S_{\mathbb{R}}^K \end{array} \quad (51)$$

commutes.

Proof. Remember that $h^K : \mathbb{S} \rightarrow S_{\mathbb{R}}^K$ is defined as the composition

$$\mathbb{S} \xrightarrow{\text{Res}_{\mathbb{C}/\mathbb{R}}(\mu^K)} \text{Res}_{\mathbb{C}/\mathbb{R}}(S_{\mathbb{C}}^K) \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} S_{\mathbb{R}}^K, \quad (52)$$

where $\mu^K : \mathbb{G}_{m,\mathbb{C}} \rightarrow T_{\mathbb{C}}^K$ is defined by $\mu^K = \pi_{\mathbb{C}}^K \circ \mu_{\tau}$ (cf. C.2). Define $h_K : \mathbb{S} \rightarrow T_{\mathbb{R}}^E$ simply by

$$\mathbb{S} \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}(\mu_{\tau})} Res_{\mathbb{C}/\mathbb{R}}(T_{\mathbb{C}}^K) \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} T_{\mathbb{R}}^K. \quad (53)$$

For proving our claim it is enough to show that the following diagram

$$\begin{array}{ccccc} \mathbb{S} & \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}(\mu_{\tau})} & Res_{\mathbb{C}/\mathbb{R}}(T_{\mathbb{C}}^K) & \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} & T_{\mathbb{R}}^K \\ & \searrow Res_{\mathbb{C}/\mathbb{R}}(\mu^K) & \downarrow Res_{\mathbb{C}/\mathbb{R}}(\pi_{\mathbb{C}}^K) & & \downarrow \pi_{\mathbb{R}}^K \\ & & Res_{\mathbb{C}/\mathbb{R}}(S_{\mathbb{C}}^K) & \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} & S_{\mathbb{R}}^K \end{array} \quad (54)$$

is everywhere commutative.

The triangle on the left is commutative, because $Res_{\mathbb{C}/\mathbb{R}}$ is a functor. Thanks to theorem A.1 it is enough to show that rectangle on the right is commutative after applying the functor X^* (cf. A.3). Since $\pi^K : T^K \rightarrow S^K$ is defined to be the inclusion $X^*(S^K) \subset X^*(T^K)$ on the level of characters (cf. C.1), we see that $X^*(Res_{\mathbb{C}/\mathbb{R}}(\pi_{\mathbb{C}}^K))$ and $X^*(\pi_{\mathbb{R}}^K)$ are inclusions as well and the commutativity follows. \square

3.2 Protagonist II: \mathcal{S}_{Sh}

The construction of the Shimura datum \mathcal{S}_{Sh} in this section goes back to Shimura [Shi00], see also [Wei]. It is of the form $\mathcal{S}_{Sh} = (GSp(V_E, \psi_E), \mathbb{H}_g^{\pm}, h_{cm})$. The symplectic \mathbb{Q} -vector space (V_E, ψ_E) is defined as follows.

Choose a finite collection of primitive CM types (E_i, Φ_i) , $1 \leq i \leq r$, such that

- i) for all i the reflex field E_i^* is contained in E , i.e. $\forall i \ E_i^* \subset E$, and
- ii) the natural map (take (119) and apply the universal property from C.1)

$$S^E \xrightarrow{\prod N_{E/E_i^*}} \prod_{i=1}^r S^{E_i^*} \xrightarrow{\prod \rho_{\Phi_i}} \prod_{i=1}^r T^{E_i} \quad (55)$$

is injective. (Proposition 1.5.1 [Wei] shows that this is allways possible.)

For every $i \in \{1, \dots, r\}$ we define a symplectic form $\psi_i : E_i \times E_i \rightarrow \mathbb{Q}$ on E_i by choosing a totally imaginary generator ξ_i of E_i (over \mathbb{Q}) and setting

$$\psi_i(x, y) = Tr_{E_i/\mathbb{Q}}(\xi_i x y^{\iota}). \quad (56)$$

Now we define (V_E, ψ_E) as the direct sum of the symplectic spaces (E_i, ψ_i) . Instead of $GSp(V_E, \psi_E)$ we will sometimes simply write GSp .

To define the morphism h_{cm} the essential step is to observe (see Remark 9.2 [Mil98]) that the image of the map $\rho_{\Phi_i} \circ N_{E/E_i^*} : S^E \rightarrow T^{E_i}$ is contained in the subtorus \mathcal{T}^{E_i} of T^{E_i} , which is defined on the level of \mathbb{Q} -points by

$$\mathcal{T}^{E_i}(\mathbb{Q}) = \{x \in E_i^{\times} \mid x x^{\iota} \in \mathbb{Q}^{\times}\} \quad (57)$$

and analogously $T^{E_i}(R)$ is defined for an arbitrary \mathbb{Q} -algebra R . This is an important observation, because there is an obvious inclusion of algebraic groups (cf. A.2)

$$\mathbf{i} : \prod_{i=1}^r \mathcal{T}^{E_i} \rightarrow GSp(V_E, \psi_E), \quad (58)$$

whereas there is in general **no** embedding $\prod T^{E_i} \rightarrow GSp$. With this in mind we define h_{cm} as the composition

$$\mathbb{S} \xrightarrow{h^E} S_{\mathbb{R}}^E \xrightarrow{\prod N_{E/E_i^*, \mathbb{R}}} \prod_{i=1}^r S_{\mathbb{R}}^{E_i^*} \xrightarrow{\prod \rho_{\Phi_i, \mathbb{R}}} \prod_{i=1}^r \mathcal{T}_{\mathbb{R}}^{E_i} \xrightarrow{\mathbf{i}_{\mathbb{R}}} GSp_{\mathbb{R}}. \quad (59)$$

Write $h'_{cm} : \mathbb{S} \rightarrow \prod_{i=1}^r \mathcal{T}_{\mathbb{R}}^{E_i}$ for the composition of the first three arrows.

Remark 12. 1) By construction h_{cm} is a CM point (cf. D.6) which is needed later to construct explicitly abelian extensions K . See 4.1.

2) Viewed as a point on the complex analytic space \mathbb{H}_g^{\pm} we write x_{cm} instead of h_{cm} . Further we denote the connected component of \mathbb{H}_g^{\pm} containing x_{cm} by \mathbb{H}_g , i.e. $x_{cm} \in \mathbb{H}_g$.

Our CM point h_{cm} enjoys the following properties

- Lemma 3.2.** 1) We have $h_{cm} = \mathbf{i} \circ \prod_{i=1}^r h_{\Phi_i}$ (see (118)).
 2) The field of definition $E(x_{cm})$ of x_{cm} is equal to the composite of the reflex fields $\tilde{E} = E_1^* \cdots E_r^* \subset E$, i.e. the associated cocharacter μ_{cm} of h_{cm} is defined over \tilde{E} (see D.6).
 3) The $GSp(\mathbb{R})$ -conjugacy classes of h_{cm} can be identified with the Siegel upper-lower half plane \mathbb{H}_g^{\pm} , for some $g \in \mathbb{N}$ depending on E .

Proof. 1) This follows immediately from (131) and (135).

2) This follows from p. 105 [Mil04] and 1).

3) For this we refer to the proof of lemma 3.11 [Wei]. □

3.3 The map $\varphi : \mathcal{S}_K \rightarrow \mathcal{S}_{Sh}$

On the level of algebraic groups $\varphi : T^K \rightarrow GSp$ is simply defined as the composition

$$T^K \xrightarrow{\pi^K} S^K \xrightarrow{N_{K/E}} S^E \xrightarrow{\prod N_{E/E_i^*}} \prod_{i=1}^r S^{E_i^*} \xrightarrow{\prod \rho_{\Phi_i}} \prod_{i=1}^r \mathcal{T}^{E_i} \xrightarrow{\mathbf{i}} GSp. \quad (60)$$

For φ being a map between \mathcal{S}_K and \mathcal{S}_{Sh} we have to check that the diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_K} & T_{\mathbb{R}}^K \\ & \searrow h_{cm} & \downarrow \varphi_{\mathbb{R}} \\ & & GSp_{\mathbb{R}} \end{array} \quad (61)$$

commutes, but this compatibility is built into the construction of h_K . Using the reflex norm (cf. B.3 and (133)) we can describe φ as follows

Lemma 3.3. *The map $\varphi : T^K \rightarrow GSp$ is equal to the composition*

$$T^K \xrightarrow{\prod N_{K/E_i^*}} \prod T^{E_i^*} \xrightarrow{\prod N_{\Phi_i}} \prod_{i=1}^r \mathcal{T}^{E_i} \xrightarrow{i} GSp. \quad (62)$$

4 About arithmetic Modular functions

4.1 Introduction

We follow closely our references [Del79], [MS81] and [Wei]. See also [Hid04]. The reader should be aware of the fact that we are using a different normalization of Artin's reciprocity map than in [MS81] and have to correct a "sign error" in [Del79] as pointed out on p. 106 [Mil04].

As usual we denote by K a number field containing a CM subfield and denote by E the maximal CM subfield of K . In this section we want to explain how the theory of Complex Multiplication provides (explicit) abelian extensions of K . In general one looks at a CM-point $x \in X$ on a Shimura variety $Sh(G, X)$ and by the theory of canonical models one knows that the point $[x, 1]$ on the canonical model $M(G, X)$ of $Sh(G, X)$ is rational over the maximal abelian extension $E(x)^{ab}$ of the field of definition $E(x)$ of x (see D.6).

In our case we look at Siegel modular varieties $Sh(GSp, \mathbb{H}_g^\pm)$ which can be considered as (fine) moduli spaces of Abelian varieties over \mathbb{C} with additional data (level structure, torsion data and polarization). See chap. 6 [Mil04] for an explanation of this. Each point $x \in \mathbb{H}_g$ corresponds to an Abelian variety A_x . In opposite to the case of imaginary quadratic fields, in general the field of definition $E(x)$ is neither contained in E nor in K . Therefore, in order to construct abelian extensions of K , we have to find an Abelian variety A_x such that

$$E(x) \subset K. \quad (63)$$

This is exactly the reason for our choice of $x_{cm} \in \mathbb{H}_g$ because we know (see 3.2 2)) that

$$E(x_{cm}) = \tilde{E} = E_1^* \cdots E_r^* \subset E \subset K. \quad (64)$$

Here x_{cm} corresponds to a product $A_{cm} = A_1 \times \cdots \times A_r$ of simple Abelian varieties A_i with complex multiplication given by E_i . This construction is the best one can do to generate abelian extensions of K using the theory of Complex Multiplication. The miracle here is again that the field of definition of A_{cm} and of its torsion points generate abelian extensions of $E(x_{cm})$. Now to obtain these abelian extensions explicitly one proceeds in complete analogy with the case of $\mathbb{Q}(i)$ explained in 2.4, namely rational functions on the connected canonical

model M° of the connected Shimura variety $Sh(GSp, \mathbb{H}_g^\pm)^\circ$ give rise to arithmetic Modular functions on \mathbb{H}_g which generate the desired abelian extensions when evaluated at x_{cm} . This will be explained in detail in the following.

4.2 Working over $\overline{\mathbb{Q}}$

4.2.1 The field \mathcal{F} of arithmetic automorphic functions

We start with the remark that the reflex field of (GSp, \mathbb{H}_g^\pm) (cf. D.6) is equal to \mathbb{Q} (see remark 32).

Remark 13. This is the second notion of "reflex field". But the reader shouldn't get confused.

Denote by Σ the set of arithmetic subgroups Γ of $GSp^{ad}(\mathbb{Q})^+$ which contain the image of a congruence subgroup of $GSp^{der}(\mathbb{Q})$. The connected component of the identity Sh° of $Sh = Sh(GSp, \mathbb{H}_g^\pm)$ is then given by the inverse limit $Sh^\circ = \varprojlim_{\Gamma \in \Sigma} \Gamma \backslash \mathbb{H}_g$ (cf. D.4).

Denote by $M^\circ = M^\circ(GSp, \mathbb{H}_g^\pm)$ the canonical model of Sh° in the sense of 2.7.10 [Del79], i.e. M° is defined over $\overline{\mathbb{Q}}$. For every $\Gamma \in \Sigma$ the space $\Gamma \backslash \mathbb{H}_g$ is an algebraic variety over \mathbb{C} and $\Gamma \backslash M^\circ$ a model over $\overline{\mathbb{Q}}$.

The field of rational functions $k(\Gamma \backslash M^\circ)$ on $\Gamma \backslash M^\circ$ is contained in the field of rational functions $k(\Gamma \backslash \mathbb{H}_g)$. Elements in the latter field correspond to meromorphic functions on \mathbb{H}_g (now viewed as a complex analytic space) that are invariant under $\Gamma \in \Sigma$.

Following [MS81] we call the field $\mathcal{F} = \bigcup_{\Gamma \in \Sigma} k(\Gamma \backslash M^\circ)$ the field of **arithmetic automorphic functions** on \mathbb{H}_g .

4.2.2 About $Aut_{\mathbb{Q}}(\mathcal{F})$

The (topological) group \mathcal{E} defined by the extension (see 2.5.9 [Del79])

$$1 \longrightarrow \overline{GSp^{ad}(\mathbb{Q})^+} \longrightarrow \mathcal{E} \xrightarrow{\sigma} Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \quad (65)$$

is acting continuously on M° (2.7.10 [Del79]) and induces an action on \mathcal{F} by

$${}^\alpha f = \sigma(\alpha) \cdot (f \circ \alpha^{-1}) = (\sigma(\alpha)f) \circ (\sigma(\alpha)\alpha^{-1}) \quad (66)$$

(see [MS81] 3.2). This is meaningful because f and α^{-1} are both defined over $\overline{\mathbb{Q}}$. Using this action one can prove

Theorem 4.1 (3.3 [MS81]). *The map $\mathcal{E} \rightarrow Aut_{\mathbb{Q}}(\mathcal{F})$ given by (66) identifies \mathcal{E} with an open subgroup of $Aut_{\mathbb{Q}}(\mathcal{F})$.*

4.3 Going down to \mathbb{Q}^{ab}

We said that M° is defined over $\overline{\mathbb{Q}}$ but it is already defined over a subfield k of \mathbb{Q}^{ab} . More precisely k is the fixed field of the kernel of the map $Gal(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} \rightarrow$

$\pi_0\pi(GSp)$ defined in 2.6.2.1 [Del79]. Therefore the action of \mathcal{E} on M^o factors through the quotient $\bar{\mathcal{E}}$ of \mathcal{E} defined by the following commutative diagram with exact rows (see 4.2 and 4.12 [MS81] or 2.5.3 [Del79])

$$\begin{array}{ccccccc}
1 & \longrightarrow & \overline{GSp^{ad}(\mathbb{Q})}^+ & \longrightarrow & \mathcal{E} & \xrightarrow{\sigma} & Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \\
& & \downarrow id & & \downarrow pr & & \downarrow res \\
1 & \longrightarrow & \overline{GSp^{ad}(\mathbb{Q})}^+ & \longrightarrow & \bar{\mathcal{E}} & \xrightarrow{\sigma} & Gal(k/\mathbb{Q}) \longrightarrow 1 \\
& & \downarrow id & & \downarrow \tau & & \downarrow l \\
1 & \longrightarrow & \overline{GSp^{ad}(\mathbb{Q})}^+ & \longrightarrow & \frac{GSp(\mathbb{A}_f)}{C(\mathbb{Q})} & \longrightarrow & \pi_0\pi(GSp) \longrightarrow 1,
\end{array} \tag{67}$$

here C denotes the center of GSp .

Remark 14. 1) We know that $C(\mathbb{Q}) = \mathbb{Q}^\times$ is discrete in $GSp(\mathbb{A}_f)$ (cf. [Hid04]).
2) Because l is injective, τ is injective as well and we can identify $\bar{\mathcal{E}}$ with an (open) subgroup of $\frac{GSp(\mathbb{A}_f)}{\mathbb{Q}^\times}$.
3) $\bar{\mathcal{E}}$ is of course depending on K but we suppress this dependence in our notation.

4.3.1 The field \mathcal{M} of arithmetic Modular functions

Let $f \in \mathcal{F}$ be a rational function, i.e. f is a rational function on $\Gamma \backslash M^o$, for some $\Gamma \in \Sigma$. We call f an **arithmetic Modular function** if it is rational over k , and meromorphic at the cusps (when viewed on the corresponding complex analytic space). Compare this to 3.4 [Wei] or pp. 35 [Mil98].

Definition 4.1. The subfield of \mathcal{F} generated by all arithmetic Modular functions is denoted by \mathcal{M} . Further we denote by \mathcal{M}^{cm} the subring of \mathcal{M} of all arithmetic Modular functions which are defined in x_{cm} .

The importance of \mathcal{M}^{cm} for our purposes is explained (see 14.4 [Mil98] and 3.11 [Wei]) by

Theorem 4.2. Let A_{cm} denote the abelian variety corresponding to x_{cm} (cf. 4.1).

Denote by $K^{A_{cm}}$ the field extension of K obtained by adjoining the field of definition of A_{cm} and all of its torsion points.

Further denote by $K^{\mathcal{M}}$ the field extension of K obtained by adjoining the values $f(x_{cm})$, for $f \in \mathcal{M}^{cm}$.

Finally denote by K^c the composition of K with the fixed field of the image of the Verlagerungsmap $Ver : Gal(F^{ab}/F) \rightarrow Gal(E^{ab}/E)$, where F is the maximal totally real subfield of E . Then we have the equality

$$K^c = K^{A_{cm}} = K^{\mathcal{M}}. \tag{68}$$

Remark 15. Notice we are not simply using the field of arithmetic automorphic functions as considered by Shimura, see 4.8 [MS81] for his definition, because the exact size of the abelian extension obtained by using these functions is not clear (at least to the author). It is clear that the field of Shimura is contained in $K^{\mathcal{M}}$ and we guess that it should generate the same extension K^c of K .

4.3.2 About $Aut_{\mathbb{Q}}(\mathcal{M})$

It is clear that \mathcal{M} is closed under the action of $\overline{\mathcal{E}}$ (see 3.2 or 4.4 [MS81]) and therefore we obtain a continuous map

$$\overline{\mathcal{E}} \rightarrow Aut_{\mathbb{Q}}(\mathcal{M}) \quad (69)$$

given like above by

$${}^{\alpha}f = \sigma(\alpha) \cdot (f \circ \alpha^{-1}) = (\sigma(\alpha)f) \circ (\sigma(\alpha)\alpha^{-1}). \quad (70)$$

In particular $GSp^{ad}(\mathbb{Q})^+$ is acting on \mathcal{M} by

$${}^{\alpha}f = f \circ \alpha^{-1}. \quad (71)$$

4.4 The reciprocity law at x_{cm}

Write

$$\mu_{cm} : \mathbb{G}_{m,\mathbb{C}} \longrightarrow S_{\mathbb{C}}^E \xrightarrow{(h'_{cm})_{\mathbb{C}}} \prod_{i=1}^r \mathcal{T}^{E_i, \mathbb{C}} \xrightarrow{i} GSp_{\mathbb{C}} \quad (72)$$

for the associated cocharacter of h_{cm} (cf. D.6). From Lemma 3.2 1) we know that $h_{cm} = i \circ \prod h_{\phi_i}$ and therefore $\mu_{cm} = i \circ \prod \mu_{\phi_i}$. Because μ_{ϕ_i} is defined over E_i^* the cocharacter $\mu'_{cm} = \prod \mu_{\phi_i}$ is defined over $\tilde{E} = E_1^* \cdots E_r^* \subset E \subset K$. To simplify the notation set $\mathcal{T} = \prod_{i=1}^r \mathcal{T}^{E_i}$. Define the morphism

$$\eta : T^K \rightarrow GSp \quad (73)$$

as composition of

$$T^K \xrightarrow{Res_{K/\mathbb{Q}}(\mu'_{cm})} Res_{K/\mathbb{Q}}(\mathcal{T}_K) \xrightarrow{Nm_{K/\mathbb{Q}}} \mathcal{T} \xrightarrow{i} GSp. \quad (74)$$

If we identify $\overline{\mathcal{E}}$ with an (open) subset of $\frac{GSp(\mathbb{A}_f)}{\mathbb{Q}^{\times}}$ using τ (see (67)), one can show (see 4.5 [MS81] or 2.6.3 [Del79]) that $\eta(\mathbb{A}) : \mathbb{A}_K^{\times} \rightarrow GSp(\mathbb{A})$ induces, by $\nu \mapsto \eta(\mathbb{A}_f)(\nu) \bmod \mathbb{Q}^{\times}$, a group homomorphism

$$\overline{\eta} : \mathbb{A}_K^{\times} \rightarrow \overline{\mathcal{E}}. \quad (75)$$

If we denote by $[\nu] \in Gal(K^{ab}/K)$ the image of $\nu \in \mathbb{A}_K^{\times}$ under Artin's reciprocity map one can show (cf. 4.5 [MS81]) that (cf. (67))

$$\sigma(\overline{\eta}(\nu)) = [\nu]^{-1}|_k. \quad (76)$$

Remark 16. The careful reader will ask why it is allowed to define η using the extension K of the field of definition of the cocharacter μ_{cm} given by \tilde{E} , because our reference [MS81] uses \tilde{E} to define η . The explanation for this is given by lemma 4.5 and standard class field theory.

Now we are able to state the **reciprocity law**

Theorem 4.3 (see 4.6 and 4.10 [MS81]). *Let $\nu \in \mathbb{A}_K^\times$ and $f \in \mathcal{M}^{cm}$. Then $f(x_{cm})$ is rational over K^{ab} . Further $\bar{\eta}(\nu)f$ is defined in x_{cm} and*

$$\bar{\eta}(\nu)f(x_{cm}) = [\nu]^{-1}(f(x_{cm})). \quad (77)$$

Proof. We simply reproduce the argument given in the proof of Thm. 4.6 [MS81]. The first assertion is clear by the definition of the canonical model (cf. D.6) and the other two assertions follow from the following calculation. Regard the special point x_{cm} as a point on the canonical model $[x_{cm}, 1] \in M^o$. The action of $\bar{\eta}(\nu)^{-1}$ is given by $\bar{\eta}(\nu)^{-1}[x_{cm}, 1] = \sigma(\bar{\eta}(\nu)^{-1})[x_{cm}, \eta(\nu)]$ and further we know $[x_{cm}, \eta(\nu)] = [\nu]^{-1}[x_{cm}, 1]$ (by (142)). Therefore we obtain

$$\begin{aligned} \bar{\eta}(\nu)f(x_{cm}) &= \sigma(\bar{\eta}(\nu)) \cdot (f \circ \bar{\eta}(\nu)^{-1})([x_{cm}, 1]) \\ &= (\sigma(\bar{\eta}(\nu))f) \circ (\sigma(\bar{\eta}(\nu))\bar{\eta}(\nu)^{-1})([x_{cm}, 1]) \\ &= (\sigma(\bar{\eta}(\nu))f) \circ (\sigma(\bar{\eta}(\nu))\sigma(\bar{\eta}(\nu))^{-1})([x_{cm}, \eta(\nu)]) \\ &\stackrel{(76)}{=} ([\nu]^{-1}|_k f)([\nu]^{-1}[x_{cm}, 1]) = [\nu]^{-1}(f([x_{cm}, 1])) \\ &= [\nu]^{-1}(f(x_{cm})). \end{aligned}$$

□

The next observation is one of the key ingredients in our construction of the arithmetic subalgebra, which we call therefore

Proposition 4.4. *The two maps of algebraic groups φ and η are equal.*

Proof. This is an immediate corollary of Prop. C.1, Lemma 3.3, the compatibility properties of the norm map and the next simple Lemma. □

Lemma 4.5. *Let (L, ϕ) a CM type and L' a finite extension of the reflex field L^* . Then the following diagram*

$$\begin{array}{ccc} T^{L'} & \xrightarrow{Res_{L'/\mathbb{Q}} \mu_{L'}} & Res_{L'/\mathbb{Q}}(T_{L'}^L) \xrightarrow{Nm_{L'/\mathbb{Q}}} T^L \\ \downarrow N_{L'/L^*} & & \nearrow Nm_{L^*/\mathbb{Q}} \\ T^{L^*} & \xrightarrow{Res_{L^*/\mathbb{Q}} \mu_{L^*}} & Res_{L^*/\mathbb{Q}}(T_{L^*}^L) \end{array} \quad (78)$$

is commutative.

Now having all the number-theoretic ingredients we need in hand, we can move on to the "operator-theoretic" part of this paper.

5 On Bost-Connes-Marcolli systems

We review very briefly the general construction of C^* -dynamical systems, named Bost-Connes-Marcolli systems, as given in [HP05].

5.1 BCM pairs

A BCM pair $(\mathcal{D}, \mathcal{L})$ is a pair consisting of a BCM datum $\mathcal{D} = (G, X, V, M)$ together with a level structure $\mathcal{L} = (L, \Gamma, \Gamma_M)$ of \mathcal{D} .

A BCM datum is a Shimura datum (G, X) together with an enveloping algebraic semigroup M and a faithful representation $\phi : G \rightarrow GL(V)$ such that $\phi(G) \subset M \subset \text{End}(V)$. Here V denotes a \mathbb{Q} -vector space of finite dimension.

A level structure \mathcal{L} of \mathcal{D} consists of a lattice $L \subset V$, a compact open subgroup $\Gamma \subset G(\mathbb{A}_f)$ and a compact open semigroup $\Gamma_M \subset M(\mathbb{A}_f)$ such that $\phi(\Gamma) \subset \Gamma_M$ and Γ_M stabilizes $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

Remark 17. In 3.1 [HP05] a more general notion of Shimura datum is allowed than ours given in the Appendix.

To every BCM datum \mathcal{D} and lattice $L \subset V$ one can associate the following so called **maximal level structure** to obtain a BCM pair by setting $\Gamma_M = M(\mathbb{A}_f) \cap \text{End}(L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$ and $\Gamma = \phi^{-1}(\Gamma_M^\times)$.

The level structure \mathcal{L} is called **fine** if Γ is acting freely on $G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f))$.

Remark 18. For the definition of the topology of $G(\mathbb{A}_f)$ and $M(\mathbb{A}_f)$ we refer the reader to [PR94]. Especially one can show that $\phi(\mathbb{A}_f) : G(\mathbb{A}_f) \rightarrow M(\mathbb{A}_f)$ is a continuous map (cf. lemma 5.2 [PR94]).

5.2 Quotient maps attached to BCM pairs

Let $(\mathcal{D}, \mathcal{L})$ a BCM pair.

5.2.1 The BCM groupoid

There is a partially defined action of $G(\mathbb{A}_f)$ on the direct product $\Gamma_M \times Sh(G, X)$ given by

$$g(\rho, [z, l]) = (g\rho, [z, lg^{-1}]), \quad (79)$$

where we suppressed the morphism ϕ . Using this the BCM groupoid U is the topological groupoid (using the notation given in (13)) defined by

$$U = G(\mathbb{A}_f) \boxplus (\Gamma_M \times Sh(G, X)). \quad (80)$$

There is an action of the group $\Gamma \times \Gamma$ on U given by

$$(\gamma_1, \gamma_2)(g, \rho, [z, l])(\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, [z, l \gamma_2^{-1}]). \quad (81)$$

We denote the quotient by $Z = (\Gamma \times \Gamma) \backslash U$ and obtain a natural quotient map

$$U \longrightarrow Z. \quad (82)$$

We denote elements in Z by $[g, \rho, [z, l]]$.

Remark 19. In general the quotient Z is not a groupoid anymore, see 4.2.1 [HP05].

5.2.2 The positive BCM groupoid

Assume that the Shimura datum (G, X) of our BCM pair satisfies (SV5) (cf. D.1). Moreover we choose a connected component X^+ of X and set $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$, where $G(\mathbb{R})^+$ denotes the connected component of the identity of $G(\mathbb{R})$. Then $G(\mathbb{Q})^+$ is acting naturally on X^+ , because X^+ can be regarded as a $G(\mathbb{R})^+$ -conjugacy class (see D.4). Now we can consider the positive (BCM) groupoid U^+ which is the topological groupoid given by

$$U^+ = G(\mathbb{Q})^+ \boxplus (\Gamma_M \times X^+). \quad (83)$$

If we set $\Gamma^+ = \Gamma \cap G(\mathbb{Q})^+$ we see further that $\Gamma^+ \times \Gamma^+$ is acting on U^+ by

$$(\gamma_1, \gamma_2)(g, \rho, z) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z). \quad (84)$$

We denote the quotient by $Z^+ = (\Gamma^+ \times \Gamma^+) \backslash U^+$ and obtain another quotient map

$$U^+ \longrightarrow Z^+. \quad (85)$$

There is a natural equivariant morphism of topological groupoids

$$U^+ \longrightarrow U \quad (86)$$

given by $(g, \rho, z) \mapsto (g, \rho, [z, 1])$, inducing a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \uparrow & & \uparrow \\ U^+ & \longrightarrow & Z^+ \end{array} \quad (87)$$

The following criterion given in 5.1 [HP05] will be crucial for our approach

Criterion 5.1. *If the natural map $G(\mathbb{Q}) \cap \Gamma \rightarrow G(\mathbb{Q})/G(\mathbb{Q})^+$ is surjective and $|G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/\Gamma| = 1$, then the natural morphism (86) induces a homeomorphism of topological spaces*

$$Z^+ \longrightarrow Z. \quad (88)$$

Remark 20. 1) In other words the two conditions of the criterion simply mean that we have a decomposition of the form $G(\mathbb{A}_f) = G(\mathbb{Q})^+ \cdot \Gamma$.

2) The inverse $Z \rightarrow Z^+$ of the above homeomorphism is given explicitly as follows. By using the first remark we can write every $l \in G(\mathbb{A}_f)$ as a product $l = \alpha\beta$ with $\alpha \in G(\mathbb{Q})^+$ and $\beta \in \Gamma$ (this decomposition is unique up to an element in $\Gamma^+ = \Gamma \cap G(\mathbb{Q})^+$). In particular every element $[g, \rho, [z, l]] \in Z$ can be written as $[g, \rho, [z, l]] = [g\beta^{-1}, \beta\rho, [\alpha^{-1}z, 1]]$ and under the inverse of the above homeomorphism this element is sent to $[g\beta^{-1}, \beta\rho, \alpha^{-1}z] \in Z^+$.

5.2.3 The adjoint BCM algebra

Let us denote by C the center of G and **assume** further that $\phi(\overline{C(\mathbb{Q})})$ is a normal subsemigroup of $M(\mathbb{A}_f)$. The adjoint group G^{ad} of G is the quotient of G by its center C (in the sense of algebraic groups, see [Wat79]). Let us define the semigroup Γ_M^{ad} to be the quotient of Γ_M by the normal subsemigroup $\phi(\overline{C(\mathbb{Q})}) \cap \Gamma_M$ and remember that X^+ can be naturally regarded as a $G^{ad}(\mathbb{R})^+$ -conjugacy class (see D.4). With this in hand we define the adjoint (BCM) groupoid U^{ad} to be the topological groupoid

$$U^{ad} = G^{ad}(\mathbb{Q})^+ \boxplus (\Gamma_M^{ad} \times X^+). \quad (89)$$

It is known that the projection $G \rightarrow G^{ad}$ induces a surjective group homomorphism $\pi^{ad} : G(\mathbb{Q})^+ \rightarrow G^{ad}(\mathbb{Q})^+$ (see 5.1 [Mil04]). Setting $\Gamma^{ad} = \pi^{ad}(\Gamma^+)$ we see immediately that $\Gamma^{ad} \times \Gamma^{ad}$ is acting on U^{ad} exactly as in 84. We obtain yet another quotient map

$$U^{ad} \longrightarrow Z^{ad}. \quad (90)$$

Using the two projections $\Gamma_M \rightarrow \Gamma_M^{ad}$ and π^{ad} there is by construction an obvious equivariant morphism of topological groupoids

$$U^+ \longrightarrow U^{ad} \quad (91)$$

which induces (together with (87)) a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \uparrow & & \uparrow \\ U^+ & \longrightarrow & Z^+ \\ \downarrow & & \downarrow \\ U^{ad} & \longrightarrow & Z^{ad}. \end{array} \quad (92)$$

5.3 BCM algebras and systems

Let $(\mathcal{D}, \mathcal{L})$ be a BCM pair. The **BCM algebra** $H = H_{\mathcal{D}, \mathcal{L}}$ is defined to be the set of compactly supported, continuous function on the quotient $Z = (\Gamma \times \Gamma) \backslash U$ of the BCM groupoid U , i.e.

$$H = C_c(Z). \quad (93)$$

By viewing functions in H as $\Gamma \times \Gamma$ -invariant functions on the groupoid U , we can equip H with the structure of a $*$ -algebra by using the usual convolution and involution on U (like in the construction of groupoid C^* -algebras). We refer to 4.3.2 [HP05] for the details. After completing H in a suitable norm we obtain a C^* -algebra A (see 6.2 [HP05]). Further there is a time evolution $(\sigma_t)_{t \in \mathbb{R}}$ on

H resp. A so that we end up with the **BCM system** $\mathcal{A} = \mathcal{A}_{\mathcal{D}, \mathcal{L}}$ given by the C^* -dynamical system

$$\mathcal{A} = (A, (\sigma_t)_{t \in \mathbb{R}}) \quad (94)$$

associated with the BCM pair $(\mathcal{L}, \mathcal{D})$. For the general definition of the time evolution we refer to 4.4 [HP05]. We will state the **time evolution** $(\sigma_t)_{t \in \mathbb{R}}$ only in the case of our BC-type systems \mathcal{A}_K (cf. 6.1.3).

Remark 21. In complete analogy one might construct a positive respectively adjoint BCM system, but we don't need this.

5.4 On Symmetries of BCM algebras

In section 4.5 [HP05] the authors define symmetries of BCM algebras but for our purpose we need to deviate from their definition in order to be in accordance with the definition of symmetries for BC-type systems given in [LLN09].

Let $(\mathcal{D}, \mathcal{L})$ be a BCM pair with fine level structure (see 5.1) and recall (D.2) that there is a natural right action of $G(\mathbb{A}_f)$ on the Shimura variety $Sh(G, X)$ which is denoted by $m[z, l] = [z, lm]$. Define the subgroup $G_\Gamma(\mathbb{A}_f) = \{g \in G(\mathbb{A}_f) \mid g\gamma = \gamma g \forall \gamma \in \Gamma\}$. Further, if we denote by C the center of G , the group $C(\mathbb{R})$ is acting on $Sh(G, X)$ by $c[z, l] = [cz, l]$. We end up with a right action of $G_\Gamma(\mathbb{A}_f) \times C(\mathbb{R})$ as symmetries on the BCM algebra $H_{\mathcal{D}, \mathcal{L}}$ given on a function $f \in C_c(Z)$ by

$$^{(m, c)}f(g, \rho, [z, l]) = f(g, \rho, [cz, lm]). \quad (95)$$

Remark 22. If $G(\mathbb{A}_f)$ is a commutative group and we have a decomposition $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot \Gamma$ then it is immediate that our symmetries agree with the ones defined in 4.5 [HP05].

6 Two BCM pairs and a map

In this section we will apply the constructions from the last section to our Shimura data $\mathcal{S}_K = (T^K, X_K, h_K)$ and $\mathcal{S}_{Sh} = (GSp(V_E, \psi_E), \mathbb{H}_g^\pm, h_{cm})$ from section 3 and show how the two resulting systems can be related.

6.1 Costume I: $(\mathcal{D}_K, \mathcal{L}_K)$ and \mathcal{A}_K

This section is valid for an arbitrary number field K .

6.1.1 The BCM groupoid $(\mathcal{D}_K, \mathcal{L}_K)$

Let us recall the BCM pair $(\mathcal{D}_K, \mathcal{L}_K)$ from 5.5 [HP05] attached to \mathcal{S}_K . It is given by

$$(\mathcal{D}_K, \mathcal{L}_K) = ((\mathcal{S}_K, K, M^K), (\mathcal{O}_K, \hat{\mathcal{O}}_K^\times, \hat{\mathcal{O}}_K)), \quad (96)$$

where the algebraic semigroup M^K is represented by the functor which assigns to a \mathbb{Q} -algebra R the semigroup of \mathbb{Q} -algebra homomorphisms $\text{Hom}(K[X], K \otimes_{\mathbb{Q}} R)$. By definition we have that $M^K(R)^{\times} = T^K(R)$ for every \mathbb{Q} -algebra R , which gives an embedding $\phi : T^K \rightarrow M^K$. (It will be convenient to set $\Gamma_K = \widehat{\mathcal{O}}_K^{\times}$.)

6.1.2 The quotient map $U_K \rightarrow Z_K$

The corresponding BCM groupoid, denoted by U_K , is given by

$$U_K = T^K(\mathbb{A}_f) \boxtimes (\widehat{\mathcal{O}}_K \times \text{Sh}(\mathcal{S}_K)) \quad (97)$$

with $\Gamma_K^2 = \widehat{\mathcal{O}}_K^{\times} \times \widehat{\mathcal{O}}_K^{\times}$ acting as in (81). We denote the quotient of this action by

$$Z_K = \Gamma_K^2 \backslash U_K. \quad (98)$$

6.1.3 The time evolution

Following 7.1 [HP05] the time evolution $(\sigma_t)_{t \in \mathbb{R}}$ on the BCM algebra $H_K = C_c(Z_K)$ is given as follows. Denote by $\mathfrak{N} = \mathfrak{N}_{K/\mathbb{Q}} : \mathbb{A}_{K,f}^{\times} \rightarrow \mathbb{R}$ the usual idele norm. Let $f \in H_K$ be a function, then we have

$$\sigma_t(f)(g, \rho, [z, l]) = \mathfrak{N}(g)^{it} f(g, \rho, [z, l]). \quad (99)$$

6.1.4 On symmetries

First, from 2.2.3 [Del79] we know that there is an isomorphism between $\text{Sh} = \text{Sh}(T^K, X_K, h_K)$ and $\pi_0(C_K)$. By class field theory the latter space $\pi_0(C_K) = C_K/D_K$ is identified with the Galois group $\text{Gal}(K^{ab}/K)$ of the maximal abelian extension of K using the Artin reciprocity homomorphism. Under this identification the natural action of $T^K(\mathbb{A}_f) = \mathbb{A}_{K,f}^{\times}$ on Sh corresponds simply to the Artin reciprocity map, i.e. if ν is a finite idele in $T(\mathbb{A}_f)$ and $\omega_1 = [g, l] \in \text{Sh}$ corresponds to the identity in $\text{Gal}(K^{ab}/K)$, then $\nu\omega_1 = [g, l\nu]$ corresponds to the image $[\nu]$ in $\text{Gal}(K^{ab}/K)$ of ν under Artin's reciprocity map.

Now, because T^K is commutative we see that $C(\mathbb{A}_f) \times C(\mathbb{R}) = T^K(\mathbb{A}) = \mathbb{A}_K^{\times}$ is acting by symmetries on H_K . By what we just said this action is simply given by the natural map $\mathbb{A}_K^{\times} \rightarrow \pi_0(C_K) = C_K/D_K$ so that we obtain (tautologically) the desired action of $C_K/D_K \cong \text{Gal}(K^{ab}/K)$ on H_K .

Remark 23. The reader should notice that in the case of an imaginary quadratic number field K our symmetries do not agree with the symmetries defined [CMR05] (except when the class number of K is equal to one, where the two definitions agree). For a short discussion on this matter we refer the reader to remark 25.

6.1.5 About extremal KMS_∞ -states of \mathcal{A}_K

We refer to pp. 445 [CM08] or [BR81] for the notion of extremal KMS_∞ -states.

Let $\mathcal{A}_K = (A_K, (\sigma_t)_{t \in \mathbb{R}})$ denote the corresponding BCM system (cf. 5.3). In Theorem 2.1 (vi) [LLN09] it is shown that the set \mathcal{E}_∞ of extremal KMS_∞ -states of \mathcal{A}_K is indexed by the set $Sh = Sh(T^K, X_K, h_K)$ and the extremal KMS_∞ -state ϱ_ω associated with $\omega \in Sh$ is given on a function $f \in H_K$ by evaluation, namely

$$\varrho_\omega(f) = f(1, 1, \omega). \quad (100)$$

Remark 24. 1) It follows immediately that the symmetry group C_K/D_K is acting free and transitively on the set of extremal KMS_∞ -states.

2) Using the definition of symmetries given in [HP05] we obtain another action of C_K/D_K on H_K but in general this will not induce a free and transitive action on the extremal KMS_∞ -states. In fact the two different actions of C_K/D_K on H_K given in [LLN09] resp. [HP05] are equivalent if and only if the class number h_K of K is equal to 1. This follows directly from the fact that if (and only if) $h_K = 1$ then already $\hat{\mathcal{O}}_K^\times$ surjects onto $Gal(K^{ab}/K)$.

All put together we have

Theorem 6.1 ([HP05] and [LLN09]). *Let K be an arbitrary number field. Then the BCM system $\mathcal{A}_K = (A_K, (\sigma_t)_{t \in \mathbb{R}})$ is a BC-type system (cf. page 2).*

6.2 Costume II: $(\mathcal{D}_{Sh}, \mathcal{L}_{Sh})$

6.2.1 The BCM groupoid $(\mathcal{D}_{Sh}, \mathcal{L}_{Sh})$

Recall the construction of the symplectic vector space (V_E, ψ_E) (see 3.2). We still have some freedom in specifying the totally imaginary generators ξ_i of the (primitive) CM fields E_i , which in turn define the symplectic form ψ_E (cf. (56)). Let us denote by L_E the lattice $L_E = \oplus_{i=1}^r \mathcal{O}_{E_i} \subset V_E = \oplus_{i=1}^r E_i$. We now fix generators ξ_i according to

Lemma 6.2. *For each $i \in \{1, \dots, r\}$ there exists a totally imaginary generator $\xi_i \in E_i$, such that the associated symplectic vector space (V_E, ψ_E) is integral with respect to L_E , i.e. there exists a symplectic basis $\{e_j\}$ for (V_E, ψ_E) such that $e_j \in L_E$, for each j .*

Proof. For each i , choose any totally imaginary generator $\tilde{\xi}_i \in E_i$ and regard the associated symplectic form $\tilde{\psi}_E$ on V_E (see (56)). It is known that there exists a symplectic basis $\{\tilde{e}_j\}$ for $(V_E, \tilde{\psi}_E)$. Now for each j there exists a $q_j \in \mathbb{N}$ such that $e_j = q_j \tilde{e}_j \in L_E$, because $E_i = \mathcal{O}_{E_i} \otimes_{\mathbb{Z}} \mathbb{Q}$. Set $q = \prod_j q_j$ and define the symplectic form ψ_E on V_E by using the totally imaginary generators $\xi_i = q^{-2} \tilde{\xi}_i \in E_i$, for every i . By construction it is now clear that $\{e_j\}$ is an integral symplectic basis for (V_E, ψ_E) . \square

Now having fixed our Shimura datum $\mathcal{S}_{Sh} = (GSp(V_E, \psi_E), \mathbb{H}_g^\pm, h_{cm})$ we define the BCM pair $(\mathcal{D}_{Sh}, \mathcal{L}_{Sh})$ equipped with the maximal level structure (cf. 5.1) with respect to the lattice L_E by

$$(\mathcal{D}_{Sh}, \mathcal{L}_{Sh}) = ((\mathcal{S}_{Sh}, V_E, MSp), (L_E, \Gamma_{Sh}, \Gamma_{Sh, M})), \quad (101)$$

where the algebraic semigroup $MSp = MSp(V_E, \psi_E)$ is represented by the functor which assigns to a \mathbb{Q} -algebra R the semigroup

$$MSp(R) = \{f \in \text{End}_R(V_E \otimes_{\mathbb{Q}} R) \mid \exists \nu(f) \in R : \psi_{E, R}(f(x), f(y)) = \nu(f) \psi_{E, R}(x, y) \ \forall x, y\}.$$

It is clear by definition (compare A.2) that $MSp(R)^\times = GSp(R)$ which defines a natural injection $\phi : GSp \rightarrow MSp$.

6.2.2 Some quotient maps

We denote the corresponding BCM groupoid by

$$U_{Sh} = GSp(\mathbb{A}_f) \boxtimes (\Gamma_{Sh, M} \times Sh(\mathcal{S}_{Sh})), \quad (102)$$

where the group $\Gamma_{Sh}^2 = \Gamma_{Sh} \times \Gamma_{Sh}$ is acting as usual. We denote the quotient of U_{Sh} by this action by

$$Z_{Sh} = \Gamma_{Sh}^2 \backslash U_{Sh}. \quad (103)$$

Thanks to D.3 and Remark 14 1) we are allowed to consider the positive and adjoint BCM groupoid, which we denote by U_{Sh}^+ and U_{Sh}^{ad} respectively. The corresponding quotients are denoted analogously by $Z_{Sh}^+ = (\Gamma_{Sh}^+)^2 \backslash U_{Sh}^+$ and $Z_{Sh}^{ad} = (\Gamma_{Sh}^{ad})^2 \backslash U_{Sh}^{ad}$.

6.3 The map $\Theta : Z_K \rightarrow Z_{Sh}^{ad}$

The aim in this section is to construct a continuous map $\Theta : Z_K \rightarrow Z_{Sh}^{ad}$.

6.3.1 Relating Z_K and Z_{Sh}

Recall that the morphism of Shimura data $\varphi : \mathcal{S}_K \rightarrow \mathcal{S}_{Sh}$ constructed in 3.3 is a morphism of algebraic groups $\varphi : T^K \rightarrow GSp$, inducing a morphism $Sh(\varphi) : Sh(\mathcal{S}_K) \rightarrow Sh(\mathcal{S}_{Sh})$ of Shimura varieties. Moreover there is a natural continuation of φ to a morphism of algebraic semigroups $M(\varphi) : M^K \rightarrow MSp$ due to the following. We know that $\varphi : T^K \rightarrow GSp$ can be expressed in terms of reflex norms (see lemma 3.3 and B.3), which are given by determinants and this definition makes still sense if we replace T^K and GSp by their enveloping semigroups M^K and MSp , respectively. Now we can define an equivariant morphism of topological groupoids

$$\Omega : U_K \rightarrow U_{Sh} \quad (104)$$

by

$$(g, m, z) \in U_K \mapsto (\varphi(\mathbb{A}_f)(g), M(\varphi)(\mathbb{A}_f)(z), Sh(\varphi)(z)) \in U_{Sh}. \quad (105)$$

To show the equivariance of Ω use $\varphi(\mathbb{A}_f)(\Gamma_K) \subset \Gamma_{Sh}$ and the equivariance of $Sh(\varphi)$ (cf. D.2). We obtain a continuous map

$$\overline{\Omega} : Z_K \rightarrow Z_{Sh}. \quad (106)$$

6.3.2 Relating Z_{Sh} and Z_{Sh}^{ad}

In order to relate Z_{Sh} and Z_{Sh}^{ad} we will show that we are allowed to apply Criterion 5.1 by proving the following two lemmata.

Lemma 6.3. *We have $GSp(V_E, \psi_E)(\mathbb{A}_f) = GSp(V_E, \psi_E)(\mathbb{Q}) \cdot \Gamma_{Sh}$.*

Proof. Let $\{e_j\}$ be an integral symplectic basis of V_E with respect to L_E (cf. Lemma 6.2). Each $f \in GSp(\mathbb{A}_f)$ is \mathbb{A}_f -linear and therefore determined by the values on $e_j \otimes 1 \otimes 1 \in L_E \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ given by

$$f(e_j \otimes 1 \otimes 1) = \sum_k a_{k,j} \otimes b_{k,j} \otimes c_{k,j} \in L_E \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}. \quad (107)$$

Let $d_{k,j} \in \mathbb{N}$ be the denominator of $b_{k,j}$ and define $c(f) = \prod_{k,j} d_{k,j} \in \mathbb{N}$. Now observe that the map

$$M_f : e_i \mapsto c(f)e_i$$

is a map in $GSp(\mathbb{Q}) \subset GSp(\mathbb{A}_f)$ i.e. M_f is compatible with the symplectic structure ψ_E . Obviously

$$M_f \circ f \in \Gamma_{Sh}$$

and thus we obtain the desired decomposition

$$f = M_f^{-1} \circ (M_f \circ f) \in GSp(\mathbb{Q}) \cdot \Gamma_{Sh}.$$

□

Lemma 6.4. *The map $GSp(\mathbb{Q}) \cap \Gamma_{Sh} \rightarrow GSp(\mathbb{Q})/GSp(\mathbb{Q})^+$ is surjective.*

Proof. We know that $GSp(\mathbb{R})^+ = \{f \in GSp(\mathbb{R}) | \nu(f) > 0\}$ (see A.2). From this we get $GSp(\mathbb{Q})^+ = GSp(\mathbb{Q}) \cap GSp(\mathbb{R})^+ = \{f \in GSp(\mathbb{Q}) | \nu(f) > 0\}$. Let f be an element in $GSp(\mathbb{Q})$. If we define M_f exactly like in the proof above, we see that $M_f \in GSp(\mathbb{Q})^+$ and can conclude $M_f \circ f \in GSp(\mathbb{Q}) \cap \Gamma_{Sh}$. □

Thus we see that the natural morphism $Z_{Sh}^+ \rightarrow Z_{Sh}$ (see 86) is a homeomorphism so that we can invert this map and compose it with the natural map $Z_{Sh}^+ \rightarrow Z_{Sh}^{ad}$ (cf. 91) to obtain a continuous map

$$Z_{Sh} \longrightarrow Z_{Sh}^{ad}. \quad (108)$$

Finally if we compose the last map with $\overline{\Omega}$ from above we obtain a continuous morphism denoted

$$\Theta : Z_K \longrightarrow Z_{Sh}^{ad}. \quad (109)$$

One crucial property of Θ is that every element $z \in Z_K$ will be sent to an element of the form

$$\Theta(z) = [g\beta^{-1}, \beta\rho, \alpha^{-1}x_{cm}] \in Z_{Sh}^{ad}, \quad (110)$$

where $g \in G^{ad}(\mathbb{Q})^+$, $\rho \in \Gamma_{Sh,M}^{ad}$, $\alpha \in G^{ad}(\mathbb{Q})$ and $\beta \in \Gamma_{Sh}^{ad}$, such that $\alpha\beta \in \pi^{ad}(\varphi(T^K(\mathbb{A}_f))) \subset G^{ad}(\mathbb{A}_f)$.

Now we are ready for the

7 Construction of our arithmetic subalgebra

The idea of the construction to follow goes back to [CMR05] and [CMR06].

We constructed the ring \mathcal{M}^{cm} of arithmetic Modular functions on \mathbb{H}_g that are defined in x_{cm} (see 4.3.1). Further the group $\overline{\mathcal{E}}$ acts by automorphisms on \mathcal{M}^{cm} according to 4.3.2. (Recall that we use the notation ${}^\rho f$ to denote the action of an automorphism α on a function $f \in \mathcal{M}^{cm}$.) Thanks to the embeddings (cf. 4.3)

$$\overline{\mathcal{E}} \longrightarrow \frac{GSp(\mathbb{A}_f)}{\mathbb{Q}^\times} \longrightarrow \frac{MSP(\mathbb{A}_f)}{\mathbb{Q}^\times} \longleftarrow \Gamma_{Sh,M}^{ad} \quad (111)$$

the intersection $\overline{\mathcal{E}} \cap \Gamma_{Sh,M}^{ad}$ is meaningful and thus we can define, for each $f \in \mathcal{M}^{cm}$, a function \tilde{f} on U_{Sh}^{ad} by

$$\tilde{f}(g, \rho, z) = \begin{cases} {}^\rho f(z), & \text{if } \rho \in \overline{\mathcal{E}} \cap \Gamma_{Sh,M}^{ad} \\ 0 & \text{else.} \end{cases} \quad (112)$$

By construction \tilde{f} is invariant under the action of $(\gamma_1, \gamma_2) \in \Gamma_{Sh}^{ad} \times \Gamma_{Sh}^{ad}$ because

$$\tilde{f}(\gamma_1 g \gamma_2^{-1}, \gamma_2 m, \gamma_2 z) \stackrel{\text{def}}{=} \gamma_2 m f(\gamma_2 z) \stackrel{(71)}{=} \gamma_2^{-1} \gamma_2 m f(z) = \tilde{f}(g, m, z). \quad (113)$$

Therefore we can regard \tilde{f} as function on the quotient $Z_{Sh}^{ad} = (\Gamma_{Sh}^{ad})^2 \backslash U_{Sh}^{ad}$.

We set $W_K = \widehat{\mathcal{O}}_K^\times \times \widehat{\mathcal{O}}_K \times Sh(T^K, X_K)$, which is a compact and clopen subset of U_K and invariant under the action of Γ_K^2 , i.e. $\Gamma_K^2 \cdot W_K \subset W_K$. With these preliminaries we have the following

Proposition 7.1. *Let f be a function in \mathcal{M}^{cm} , then $\tilde{f} \circ \Theta$ is contained in $C_c(Z_K)$, i.e.*

$$\tilde{f} \circ \Theta \in H_K \subset A_K. \quad (114)$$

Proof. As we already remarked in (110) the image of an element $z \in Z_K$ under Θ is of the form $[g\beta^{-1}, \beta\rho, \alpha^{-1}x_{cm}] \in Z_{Sh}^{ad}$. Therefore $f_K = \tilde{f} \circ \Theta$ is continuous, because Θ is continuous, the action of \mathcal{E} is continuous and does not produce singularities at special points (see Theorem 4.3).

Let us now regard f_K as Γ_K^2 -invariant function on U_K . Thanks to $\overline{\mathcal{E}} \subset \frac{GSp(\mathbb{A}_f)}{\mathbb{Q}^\times}$ and (112) we see that the support of our function f_K is contained in the clopen subset $\widehat{\mathcal{O}}_K^\times \times \widehat{\mathcal{O}}_K^\times \times Sh(T^K, X_K) \subset U_K$. Using the compact subset $W_K \subset U_K$, the next easy lemma finishes the proof. \square

Lemma 7.2. *Let G be a topological group, X be a topological G -space and $Y \subset X$ a compact, clopen subset such that $GY \subset Y$. If we have a continuous, G -invariant function $f \in C^G(X)$ then $f|_Y \in C_c(G \backslash X)$.*

Now we can define our arithmetic subalgebra of $\mathcal{A}_K = (A_K, \sigma_t)$ (cf. 6.1).

Definition 7.1. Denote by A_K^{arith} the K -rational subalgebra of A_K generated by the set of functions $\{\tilde{f} \circ \Theta \mid f \in \mathcal{M}^{cm}\}$.

7.1 Proof of Theorem 1.1

Let $f \in \mathcal{M}^{cm}$ and denote $f_K = \tilde{f} \circ \Theta \in A_K^{arith}$. Further denote by ϱ_ω the extremal KMS_∞ -state of \mathcal{A}_K corresponding to $\omega \in Sh = Sh(T^K, X_K, h_K)$ (see 6.1.5). Recall the isomorphism $Gal(K^{ab}/K) \cong \pi_0(C_K) = Sh$ given by Artin reciprocity. Considered as element in $Gal(K^{ab}/K)$ we write $[\omega]$ for ω .

Property vi)

Let $\nu \in \mathbb{A}_K^\times$ be a symmetry of \mathcal{A}_K (see 6.1.4). Thanks to Lemma 6.2 and 6.3, we can write $\varphi(\mathbb{A}_f)(\nu) = \alpha\beta \in GSp(\mathbb{A}_f)$ with $\alpha \in GSp(\mathbb{Q})^+$ and $\beta \in \Gamma_{Sh}$. By $\overline{\alpha}$ resp. $\overline{\beta}$ we will denote their images in $GSp^{ad}(\mathbb{Q})^+$ resp. Γ_{Sh}^{ad} under the map $\pi^{ad} \circ \varphi(\mathbb{A}_f)$. Moreover we denote the image of ν under Artin reciprocity by $[\nu] \in Gal(K^{ab}/K)$.

The action of the symmetries on the extremal KMS_∞ -states is given by pull back and because this action is free and transitive it is enough to restrict to the case of the extremal KMS_∞ -state ϱ_1 corresponding to the identity in $Gal(K^{ab}/K)$.

Using Proposition 4.4 and the reciprocity law (77) we can calculate the action of ν on $\varrho_\omega(f_K)$ as follows

$$\begin{aligned} \nu \varrho_1(f_K) &\stackrel{\text{def}}{=} \varrho_1(\nu f_K) \stackrel{(110)}{=} \tilde{f}(\overline{\beta}^{-1}, \overline{\beta}, \overline{\alpha}^{-1}x_{cm}) \stackrel{4.4}{=} \overline{\eta}^{(\beta)} f(\overline{\alpha}^{-1}x_{cm}) \\ &\stackrel{(71)}{=} \overline{\eta}^{(\alpha)} \overline{\eta}^{(\beta)} f(x_{cm}) = \overline{\eta}^{(\nu)} f(x_{cm}) \stackrel{(77)}{=} [\nu]^{-1}(f(x_{cm})) \\ &= [\nu]^{-1}(\varrho_1(f_K)) \end{aligned}$$

This is precisely the intertwining property we wanted to prove.

Property v)

Using the notation from above, we conclude immediately that by construction and Theorem 4.2 we have

$$\varrho_1(f_K) = \tilde{f}(1, 1, x_{xm}) = f(x_{cm}) \in K^{ab} \quad (115)$$

and the above calculation shows further that

$$\varrho_\omega(f_K) = {}^{[\omega]}\varrho_1(f_K) = [\omega]^{-1}(f(x_{cm})) \in K^{ab} \quad (116)$$

finishing our proof.

Remark 25. We want to conclude our paper with a short discussion comparing our construction with the original construction of Connes, Marcolli and Ramachdran (see [CMR05]) in the case of an imaginary quadratic field K . Apart from the fact that we are not dealing with the " K -lattice" picture as done in [CMR05] the main difference lies in the different definitions of symmetries. If the class number h_K of K is equal to one, it is immediate that the two definitions agree, however for $h_K > 1$ their symmetries contain endomorphisms (see Prop. 2.17 [CMR05]) whereas our symmetries are always given by automorphisms. We want to mention that it is no problem to generalize (this is already contained in [CM08]) their definition to the context of a BC-type system for an arbitrary number field and, without changing the definition of our arithmetic subalgebra, we could have proved Theorem 1.1 by using the new definition of symmetries (now containing endomorphisms).

This might look odd at first sight but in the context of endomotives (see [CM08] for a reference) the relationship between the two different definitions will become transparent. More precisely in a forthcoming article we will show that every BC-type system (for an arbitrary number field) is an endomotive and the precise relationship between the two definitions of symmetries (and their actions on (extremal) KMS_β -states) will become clear.

A Algebraic Groups

Our references are [Wat79] and [Mil06]. Let k denote a field of characteristic zero, K a finite field extension of k and \bar{k} an algebraic closure of k . Further we denote by R a k -algebra.

A.1 Functorial definition and basic constructions

An **(affine) algebraic group** G (over k) is a representable functor from (commutative, unital) k -algebras to groups. We denote by $k[G]$ its representing algebra, i.e. for any R we have $G(R) = \text{Hom}_{k\text{-alg}}(k[G], R)$.

A **homomorphism** $F : G \rightarrow H$ between two algebraic groups G and H (over k) is given by a natural transformation of functors.

Let G and H be two algebraic groups over k , then their **direct product** $G \times H$ is the algebraic group (over k) given by $R \mapsto G(R) \times H(R)$.

Let G be an algebraic group over k and K an extension of k . Then by **extension of scalars** we obtain an algebraic group G_K over K represented by $K \otimes_k k[G]$.

Now let G be an algebraic group over K . The **Weil restriction** $Res_{K/k}(G)$ is an algebraic group over k defined by $Res_{K/k}(G)(R) = G(K \otimes_k R)$.

Remark 26. All three constructions are functorial.

A.2 Examples

1) The **multiplicative group** $\mathbb{G}_{m,k}$ (over k) is represented by $k[x, x^{-1}] = k[x, y]/(xy - 1)$, i.e. $\mathbb{G}_{m,k}(R) = R^\times$.

2) Define $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. We have $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$. In particular we have $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$.

3) More generally an algebraic group T over k is called a **torus** if $T_{\bar{k}}$ is isomorphic to a product of copies of $\mathbb{G}_{m,\bar{k}}$.

4) The **general symplectic group** $GSp(V, \psi)$ attached to a symplectic \mathbb{Q} -vector space (V, ψ) is an algebraic group over \mathbb{Q} defined on a \mathbb{Q} -algebra R by

$$GSp(R) =$$

$$\{f \in End_R(V \otimes_{\mathbb{Q}} R) \mid \exists \nu(f) \in R^\times : \psi_R(f(x), f(y)) = \nu(f)\psi_R(x, y) \forall x, y \in V \otimes_{\mathbb{Q}} R\}.$$

A.3 Characters

Let G be an algebraic group over k and set $\Lambda = \mathbb{Z}[Gal(\bar{k}/k)]$. The **character group** $X^*(G)$ of G is defined by $Hom(G_{\bar{k}}, \mathbb{G}_{m,\bar{k}})$. There is a natural action of $Gal(\bar{k}/k)$ on $X^*(G)$, i.e. $X^*(G)$ is a Λ -module. Analogously the **cocharacter group** $X_*(G)$ of G is the Λ -module $Hom(\mathbb{G}_{m,\bar{k}}, G_{\bar{k}})$. We denote the action of $\sigma \in \Lambda$ on a (co)character f by σf or f^σ . There is the following important

Theorem A.1 (7.3 [Wat79]). *The functor $G \mapsto X^*(G)$ is a contravariant equivalence from the category of algebraic groups of multiplicative type over k and the category of finitely generated abelian groups with a continuous action of $Gal(\bar{k}/k)$.*

Remark 27. See 7.2 [Wat79] for the definition of groups of multiplicative type. We only have to know that algebraic tori are of multiplicative type.

There is a natural bi-additive and $Gal(\bar{k}/k)$ -invariant pairing $\langle \cdot, \cdot \rangle : X_*(G) \times X^*(G) \rightarrow \mathbb{Z}$ given by $\langle \chi, \mu \rangle = \mu \circ \chi \in Hom(\mathbb{G}_{m,\bar{k}}, \mathbb{G}_{m,\bar{k}}) \cong \mathbb{Z}$. If G is of multiplicative type the pairing is perfect, i.e. there is an isomorphism of Λ -modules $X_*(G) \cong Hom(X^*(G), \mathbb{Z})$.

A.4 Norm maps

Let L be a finite field extension of K , i.e. we have a tower $k \subset K \subset L$, and let T be a torus over k . Then there are two types of morphisms of algebraic groups

which we call **norm maps**. The first one

$$Nm_{L/K} : Res_{L/K}(T_L) \rightarrow T_K \quad (117)$$

is induced by the usual norm map of algebras $R \otimes_K L \rightarrow R$, for R a K -algebra. In applying the Weil restriction functor $Res_{K/k}$ we obtain the second one, namely

$$N_{L/K} : Res_{L/k}(T_L) \rightarrow Res_{K/k}(T_K).$$

A.5 The case of number fields

Let K be a number field. We are interested in the algebraic group $T^K = Res_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$ (over \mathbb{Q}). We have $T^K(R) = (K \otimes_{\mathbb{Q}} R)^{\times}$.

It is easy to see that the isomorphism $K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \prod_{\rho \in Hom(K, \overline{\mathbb{Q}})} \overline{\mathbb{Q}}$ induces an isomorphism of algebraic groups $T_{\mathbb{Q}}^K \cong \prod_{\rho \in Hom(K, \overline{\mathbb{Q}})} \mathbb{G}_{m, \overline{\mathbb{Q}}}$. It follows that $X^*(T^K) \cong \mathbb{Z}^{Hom(K, \overline{\mathbb{Q}})}$ with $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acting as follows.

For $f = \sum_{\rho \in Hom(K, \overline{\mathbb{Q}})} a_{\rho}[\rho] \in \mathbb{Z}^{Hom(K, \overline{\mathbb{Q}})}$ and $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have

$$\sigma f = \sum_{\rho \in Hom(K, \overline{\mathbb{Q}})} a_{\rho}[\sigma \circ \rho] = \sum_{\rho \in Hom(K, \overline{\mathbb{Q}})} a_{\sigma^{-1} \circ \rho}[\rho].$$

For any inclusion $K \subset L$ of number fields the norm map $N_{L/K} : T^L \rightarrow T^K$ is defined by saying that a character $f = \sum_{\rho \in Hom(K, \overline{\mathbb{Q}})} a_{\rho}[\rho] \in X^*(T^K)$ is mapped to the character $f_L = \sum_{\rho' \in Hom(L, \overline{\mathbb{Q}})} (a_{\rho'|K})[\rho'] \in X^*(T^L)$.

B CM fields

We follow [Mil06] and [Mil04]. By ι we denote the complex conjugation of \mathbb{C} .

B.1 CM fields and CM types

Let E denote a number field. If E is a totally imaginary quadratic extension of a totally real field, we call E a **CM field**. In particular the degree of a CM field is always even. A **CM type** (E, Φ) is a CM field E together with a subset $\Phi \subset Hom(E, \mathbb{C})$ such that $\Phi \cup \iota\Phi = Hom(E, \mathbb{C})$ and $\Phi \cap \iota\Phi = \emptyset$.

B.2 About h_{ϕ} and μ_{ϕ}

Let (E, Φ) be a CM type. Then there are natural isomorphisms $T_{\mathbb{R}}^E \cong \prod_{\phi \in \Phi} \mathbb{S}$ resp. $T_{\mathbb{C}}^E \cong \prod_{\phi \in \Phi} \mathbb{G}_{m, \mathbb{C}} \times \prod_{\bar{\phi} \in \iota\Phi} \mathbb{G}_{m, \mathbb{C}}$, where the first one is induced by $E \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\phi \in \Phi} \mathbb{C}$ and the second one by $E \otimes \mathbb{C} \cong \prod_{\phi \in \Phi} \mathbb{C} \times \prod_{\bar{\phi} \in \iota\Phi} \mathbb{C}$. Thus we obtain natural morphisms

$$h_{\phi} : \mathbb{S} \rightarrow T_{\mathbb{R}}^E ; z \mapsto (z)_{\phi \in \Phi} \quad (118)$$

and

$$\mu_\Phi : \mathbb{G}_{m,\mathbb{C}} \rightarrow T_{\mathbb{C}}^E ; z \mapsto (z)_{\phi \in \Phi} \times (1)_{\overline{\phi} \in \iota\Phi}. \quad (119)$$

If we take μ_Φ for granted we could have defined h_Φ by the composition

$$Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}(\mu_\Phi)} Res_{\mathbb{C}/\mathbb{R}}(T_{\mathbb{C}}^E) \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} T_{\mathbb{R}}^E. \quad (120)$$

In particular we see that h_Φ and μ_Φ are related by

$$h_{\Phi,\mathbb{C}}(z, 1) = \mu_\Phi(z). \quad (121)$$

Remark 28. In the last two sections one might have replaced \mathbb{C} by $\overline{\mathbb{Q}}$.

B.3 The reflex field and reflex norm

Let (E, Φ) be a CM type. The **reflex field** E^* of (E, Φ) is the subfield of $\overline{\mathbb{Q}}$ defined by any one of the following conditions:

- a) $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ fixes E^* if and only if $\sigma\Phi = \Phi$;
- b) E^* is the field generated over \mathbb{Q} by the elements $\sum_{\phi \in \Phi} \phi(e)$, $e \in E$;
- c) E^* is the smallest subfield of $\overline{\mathbb{Q}}$ such that there exists a $E \otimes_{\mathbb{Q}} E^*$ -module V such that

$$Tr_{E^*}(e|V) = \sum_{\phi \in \Phi} \phi(e), \text{ for all } e \in E. \quad (122)$$

The **reflex norm** of (E, Φ) is the morphism of algebraic groups $N_\Phi : T^{E^*} \rightarrow T^E$ given, for R a \mathbb{Q} -algebra, by

$$a \in T^{E^*}(R) \mapsto det_{E \otimes_{\mathbb{Q}} R}(a|V \otimes_{\mathbb{Q}} R) \in T^E(R). \quad (123)$$

C The Serre group

Our references are [Mil98], [Mil06] and [Wei]. Let K be a number field. We fix an embedding $\tau : K \rightarrow \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ and denote by ι complex conjugation on \mathbb{C} .

C.1 Definition of the Serre group

The following are equivalent:

- (1) The **Serre group** attached to K is a pair (S^K, μ^K) consisting of a \mathbb{Q} -algebraic torus S^K and a cocharacter $\mu^K \in X_*(S^K)$ defined by the following universal property. For every pair (T, μ) consisting of a \mathbb{Q} -algebraic torus T and a cocharacter $\mu \in X_*(T)$ defined over K satisfying the Serre condition

$$(\iota + 1)(\sigma - 1)\mu = 0 = (\sigma - 1)(\iota + 1)\mu \quad \forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \quad (124)$$

there exists a unique morphism $\rho_\mu : S^K \rightarrow T$ such that the diagram

$$\begin{array}{ccc} S_{\overline{\mathbb{Q}}}^K & \xrightarrow{\rho_{\mu, \overline{\mathbb{Q}}}} & T_{\overline{\mathbb{Q}}} \\ & \swarrow \mu^K \quad \searrow \mu & \\ & \mathbb{G}_{m, \overline{\mathbb{Q}}} & \end{array}$$

commutes.

(2) The **Serre group** S^K is defined to be the quotient of T^K such that $X^*(S^K)$ is the subgroup of $X^*(T^K)$ given by all elements $f \in X^*(T^K)$ which satisfy the Serre condition

$$(\sigma - 1)(\iota + 1)f = 0 = (\iota + 1)(\sigma - 1)f \quad \forall \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

The cocharacter μ^K is induced by the cocharacter $\mu_\tau \in X_*(T^K)$ defined by

$$\langle \mu_\tau, \Sigma n_\sigma[\sigma] \rangle = n_\tau, \quad \forall \Sigma n_\sigma[\sigma] \in \mathbb{Z}^{\text{Hom}(K, \overline{\mathbb{Q}})} \cong X^*(T^K).$$

(3) If K does not contain a CM subfield, we set $E = \mathbb{Q}$, otherwise E denotes the maximal CM subfield of K and F the maximal totally real subfield of E . Then there is an exact sequence of \mathbb{Q} -algebraic groups

$$1 \longrightarrow \ker(N_{F/\mathbb{Q}} : T^F \rightarrow T^{\mathbb{Q}}) \xrightarrow{i} T^K \xrightarrow{\pi^K} S^K \longrightarrow 1, \quad (125)$$

where i is the obvious inclusion. The cocharacter μ_τ of T^K , defined as in (2), induces μ^K , i.e. $\mu^K = \pi^K \circ \mu_\tau$.

Remark 29. For $K = \mathbb{Q}$ or K an imaginary quadratic field there is the obvious equality $S^K = T^K$.

C.2 About μ^K and h^K

The cocharacter $\mu^K = \pi^K \circ \mu_\tau : \mathbb{G}_{m, \mathbb{C}} \rightarrow S_{\mathbb{C}}^K$ from the last section induces a natural morphism

$$h^K : \mathbb{S} \rightarrow S_{\mathbb{R}}^K \quad (126)$$

defined by

$$\text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m, \mathbb{C}}) \xrightarrow{\text{Res}_{\mathbb{C}/\mathbb{R}}(\mu^K)} \text{Res}_{\mathbb{C}/\mathbb{R}}(S_{\mathbb{C}}^K) \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} S_{\mathbb{R}}^K. \quad (127)$$

We see that μ^K and h^K are related by

$$h_{\mathbb{C}}^K(z, 1) = \mu^K(z) \quad (128)$$

or in other words, for $z \in \mathbb{C}^\times$, we have $h^K(z) = \mu^K(z)\mu^K(z)^\iota$.

C.3 About ρ_Φ and the reflex norm N_Φ

Let (E, Φ) be a CM type. The natural morphism $\mu_\Phi \in X_*(T^E)$ (cf. 119) is defined over the reflex field E^* and an easy calculation shows that it satisfies the Serre condition (124). By the universal property of the Serre group we obtain a \mathbb{Q} -rational morphism

$$\rho_\Phi : S^{E^*} \longrightarrow T^E \quad (129)$$

such that

$$\mu_\Phi = \rho_{\Phi, \mathbb{C}} \circ \mu^{E^*}. \quad (130)$$

Also, we see immediately that

$$h_\Phi = \rho_{\Phi, \mathbb{R}} \circ h^{E^*}. \quad (131)$$

Moreover we can relate ρ_Φ and μ_Φ by the following commutative diagram

$$\begin{array}{ccccc} T^{E^*} & \xrightarrow{\text{Res}(\mu_{\Phi, E^*})} & \text{Res}_{E^*/\mathbb{Q}}(T_{E^*}^E) & \xrightarrow{N_{E^*/\mathbb{Q}}} & T^E \\ & \searrow \pi^{E^*} & & \nearrow \rho_\Phi & \\ & & S^{E^*} & & \end{array} \quad (132)$$

which can be seen on the level of characters. The relation with the reflex norm $N_\Phi : T^{E^*} \rightarrow T^E$ (see B.3) is given by the following important

Proposition C.1 ([Mil06]). *We have the equality*

$$N_\Phi = N_{E^*/\mathbb{Q}} \circ \text{Res}(\mu_{\Phi, E^*}). \quad (133)$$

C.4 More properties of the Serre group

The following properties are all taken from [Mil06].

Proposition C.2. *Let $E \subset K$ denote two number fields.*

1. *The norm map $N_{K/E} : T^K \rightarrow T^E$ induces a commutative diagram*

$$\begin{array}{ccc} T^K & \xrightarrow{N_{K/E}} & T^E \\ \downarrow \pi^K & & \downarrow \pi^E \\ S^K & \longrightarrow & S^E. \end{array} \quad (134)$$

We call the induced morphism $N_{K/E} : S^K \rightarrow S^E$.

2. There is a commutative diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h^K} & S^K \\ & \searrow h^E & \downarrow N_{K/E} \\ & & S^E \end{array} \quad (135)$$

3. Let E denote the maximal CM field contained in K , if there is no such subfield we set $E = \mathbb{Q}$. Then $N_{K/E} : S^K \rightarrow S^E$ is an isomorphism.
4. Let (E, Φ) be a CM type and $K_1 \subset K_2$ two number fields, such that $E^* \subset K_1$, and let $\rho_{\Phi,i} : S^{K_i} \rightarrow T^E$ be the corresponding maps from the universal property of the Serre group. Then we have

$$\rho_{\Phi,1} \circ N_{K_2/K_1} = \rho_{\Phi,2}.$$

D Shimura Varieties

Our references are Deligne's foundational [Del79], Milne's [Mil04] and Hida's [Hid04].

Let G be an algebraic group over \mathbb{Q} . Then the adjoint group G^{ad} of G is defined to be the quotient of G by its center C . The derived group G^{der} of G is defined to be the intersection of the normal algebraic subgroups of G such that G/N is commutative. By $G(\mathbb{R})^+$ we denote the identity component of $G(\mathbb{R})$ relative to its real topology and set $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. If G is reductive, we denote by $G(\mathbb{R})_+$ the group of elements of $G(\mathbb{R})$ whose image in $G^{ad}(\mathbb{R})$ lies in its identity component and set $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

D.1 Shimura datum

A **Shimura datum** is a pair (G, X) consisting of a reductive group G (over \mathbb{Q}) and a $G(\mathbb{R})$ -conjugacy class X of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$, such that the following (three) axioms are satisfied

- (SV1): For each $h \in X$, the representation $Lie(G_{\mathbb{R}})$ defined by h is of type $\{(-1, 1), (0, 0), (1, -1)\}$.
- (SV2): For each $h \in X$, $ad(h(i))$ is a Cartan involution on $G_{\mathbb{R}}^{ad}$.
- (SV3): G^{ad} has no \mathbb{Q} -factors on which the projection of h is trivial.

Because $G(\mathbb{R})$ is acting transitively on X it is enough to give a morphism $h_0 : \mathbb{S} \rightarrow G_{\mathbb{R}}$ to specify a Shimura datum. Therefore a **Shimura datum** is sometimes written as triple (G, X, h_0) or simply by (G, h_0) .

Further in our case of interest the following axioms are satisfied and (simplify the situation enormously).

- (SV4): The weight homomorphism $\omega_X : \mathbb{G}_{m, \mathbb{R}} \rightarrow G_{\mathbb{R}}$ is defined over \mathbb{Q} .
- (SV5): The group $C(\mathbb{Q})$ is discrete in $C(\mathbb{A}_f)$.
- (SV6): The identity component of the center C^o splits over a CM-field.
- (SC): The derived group G^{der} is simply connected.
- (CT): The center C is a cohomologically trivial torus.

Remark 30. 1) Axioms (SV1 – 6) are taken from [Mil04], the other two axioms are taken from [Hid04].
 2) The axioms of a Shimura variety (SV1-3) imply, for example, that X is a finite union of hermitian symmetric domains. When viewed as an analytic space we sometimes write x instead of h for points in X and h_x for the associated morphism $h_x : \mathbb{S} \rightarrow G_{\mathbb{R}}$.
 3) In 3.1 [HP05] a more general definition of a Shimura datum is given. For our purpose Deligne's original definition, as given above, and so called 0-dimensional Shimura varieties are sufficient.

A **morphism of Shimura data** $(G, X) \rightarrow (G', X')$ is a morphism $G \rightarrow G'$ of algebraic groups which induces a map $X \rightarrow X'$.

D.2 Shimura varieties

Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Set $Sh_K = Sh_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$, where $G(\mathbb{Q})$ is acting on X and $G(\mathbb{A}_f)$ on the left, and K is acting on $G(\mathbb{A}_f)$ on the right. One can show (see 5.13 [Mil04]) that there is a homeomorphism $Sh_K \cong \bigsqcup \Gamma_g \backslash X^+$. Here X^+ is a connected component of X and Γ_g is the subgroup $gKg^{-1} \cap G(\mathbb{Q})_+$ where g runs over a set of representatives of $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f) / K$. When K is chosen sufficiently small, then $\Gamma_g \backslash X^+$ is an arithmetic locally symmetric variety. For an inclusion $K' \subset K$ we obtain a natural map $Sh_{K'} \rightarrow Sh_K$ and in this way an inverse system $(Sh_K)_K$. There is a natural right action of $G(\mathbb{A}_f)$ on this system (cf. p 55 [Mil04]).

The **Shimura variety** $Sh(G, X)$ associated with the Shimura datum (G, X) is defined to be the inverse limit of varieties $\varprojlim_K Sh_K(G, X)$ together with the natural action of $G(\mathbb{A}_f)$. Here K runs through sufficiently small compact open subgroups of $G(\mathbb{A}_f)$. $Sh(G, X)$ can be regarded as a scheme over \mathbb{C} .

Let (G, X) be a Shimura datum such that (SV5) holds, then one has

$$Sh(G, X) = \varprojlim_K Sh_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f). \quad (136)$$

In this case we write $[x, l]$ for an element in $Sh(G, X)$ and the (right) action of an element $g \in G(\mathbb{A}_f)$ is given by

$$g[x, l] = [x, lg]. \quad (137)$$

In the general case, wenn (SV5) is not holding we use the same notation, understanding that $[x, l]$ stands for a family $(x_K, l_K)_K$ indexed by compact open subgroups K of $G(\mathbb{A}_f)$.

A **morphism of Shimura varieties** $Sh(G, X) \rightarrow Sh(G', X')$ is an inverse system of regular maps of algebraic varieties compatible with the action of $G(\mathbb{A}_f)$. We have the following functorial property:

A morphism $\varphi : (G, X) \rightarrow (G', X')$ of Shimura data defines an equivariant morphism $Sh(\varphi) : Sh(G, X) \rightarrow Sh(G', X')$ of Shimura varieties, which is a closed immersion if $G \rightarrow G'$ is injective (Thm 5.16 [Mil04]).

D.3 Example

We want to give some details about the Shimura varieties attached to the data \mathcal{S}_{Sh} constructed in 3.2. For the identification of the $GSp(\mathbb{R})$ -conjugacy class of h_{cm} with the higher Siegel upper lower half space

$$\mathbb{H}_g^\pm = \{M = A + iB \in M_g(\mathbb{C}) \mid A = A^t, B \text{ positive or negative definite} \}$$

we refer further to exercise 6.2 [Mil04].

In addition the data \mathcal{S}_{Sh} fulfill all the axioms stated in D.1. The validity of (SV1-6) is shown on p. 67 [Mil04] and the validity of (SC) and (CT) in [Hid04]. The latter two axioms are important for making the arguments in [MS81] in this case. From (SV5) follows in particular that we don't have to bother about the limits in the definition of $Sh(GSp, \mathbb{H}_g^\pm)$ because we have

$$Sh(GSp, \mathbb{H}_g^\pm) = GSp(\mathbb{Q}) \backslash (\mathbb{H}_g^\pm \times GSp(\mathbb{A}_f)).$$

D.4 Connected Shimura varieties

A **connected Shimura datum** is a pair (G, X^+) consisting of a semisimple algebraic group G over \mathbb{Q} and a $G^{ad}(\mathbb{R})^+$ -conjugacy class of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}^{ad}$ satisfying axioms (SV1-3).

The **connected Shimura variety** $Sh^o(G, X^+)$ associated with a connected Shimura datum (G, X^+) is defined by the inverse limit

$$Sh^o(G, X^+) = \varprojlim_{\Gamma} \Gamma \backslash X^+ \tag{138}$$

where Γ runs over the torsion-free arithmetic subgroups of $G^{ad}(\mathbb{Q})^+$ whose inverse image in $G(\mathbb{Q})^+$ is a congruence subgroup.

If we start with a Shimura datum (G, X) and choose a connected component X^+ of X , we can view X^+ as a $G^{ad}(\mathbb{R})^+$ -conjugacy class of morphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}^{ad}$ by projecting elements in X^+ to $G_{\mathbb{R}}^{ad}$. One can show that (G^{der}, X^+) is a connected Shimura datum. Further if we choose the connected component $Sh(G, X)^o$ of $Sh(G, X)$ containing $X^+ \times 1$, one has the following compatibility relation

$$Sh(G, X)^o = Sh^o(G^{der}, X^+). \tag{139}$$

D.5 0-dimensional Shimura varieties

In section 3.1 we defined a "Shimura datum" $\mathcal{S}_K = (T^K, X_K)$ which is not a Shimura datum in the above sense because X_K has more than one conjugacy class (recall that T^K is commutative). Rather \mathcal{S}_K is a Shimura datum in the generalized sense of Pink [Pin90] which we don't want to recall here. Instead we define the notion of a 0-dimensional Shimura varieties following [Mil04] which covers all exceptional Shimura data we consider. We define a **0-dimensional Shimura datum** to be a triple (T, Y, h) , where T is a torus over \mathbb{Q} , Y a finite set on which $T(\mathbb{R})/T(\mathbb{R})^+$ acts transitively and $h : \mathbb{S} \rightarrow T_{\mathbb{R}}$ a morphism of algebraic groups. We view Y as a finite cover of $\{h\}$. We remark that the axioms (SV1 – 3) are automatically satisfied in this setup.

The associated **0-dimensional Shimura variety** $Sh(T, Y, h)$ is defined to be the inverse system of finite sets $T(\mathbb{Q}) \backslash Y \times T(\mathbb{A}_f)/K$ with K running over the compact open subgroups of $T(\mathbb{A}_f)$.

A **morphism** $(T, Y, h) \rightarrow (H, h_0)$ from a 0-dimensional Shimura datum to a Shimura datum, with H an algebraic torus, is given by a morphism of algebraic groups $\varphi : T \rightarrow H$ such that $h = \varphi_{\mathbb{R}} \circ h_0$.

If φ is such a morphism it defines a morphism $Sh(\varphi) : Sh(T, Y, h) \rightarrow Sh(H, h_0)$ of Shimura varieties.

Remark 31. We have that \mathcal{S}_K fulfills axiom (SV5) if and only if $K = \mathbb{Q}$ or K an imaginary quadratic field (see 3.2 [HP05]).

D.6 Canonical model of Shimura varieties

Let (G, X) be a Shimura datum. A point $x \in X$ is called a **special point** if there exists a torus $T \subset G$ such that h_x factors through $T_{\mathbb{R}}$. The pair (T, x) or (T, h_x) is called special pair. If (G, X) satisfies the axioms (SV4) and (SV6), then a special point is called **CM point** and a special pair is called CM pair.

Now given a special pair (x, T) we can consider the cocharacter μ_x of $G_{\mathbb{C}}$ defined by $\mu_x(z) = h_{x, \mathbb{C}}(z, 1)$. Denote by $E(x)$ the field of definition of μ_x , i.e. the smallest subfield k of \mathbb{C} such that $\mu_x : \mathbb{G}_{m, k} \rightarrow G_k$ is defined.

Let R_x denote the composition

$$T^{E(x)} \xrightarrow{Res_{E(x)/\mathbb{Q}}(\mu_x)} Res_{E(x)/\mathbb{Q}}(T_E(x)) \xrightarrow{Nm_{E(x)/\mathbb{Q}}} T \quad (140)$$

and define the **reciprocity morphism**

$$r_x = R_x(\mathbb{A}_f) : \mathbb{A}_{E(x), f}^{\times} \rightarrow T(\mathbb{A}_f). \quad (141)$$

Moreover every datum (G, X) defines an algebraic number field $E(G, X)$, the reflex field of (G, X) . For the definition we refer the reader to 12.2 [Mil04].

Remark 32. 1) For the Shimura datum $\mathcal{S}_{Sh} = (GSp, \mathbb{H}_g^{\pm})$ (see 3.2) we have $E(\mathcal{S}_K) = \mathbb{Q}$ (cf. p. 112 [Mil04]).

2) For explanations to relations with the reflex field of a CM field (cf. B.3), see example 12.4 b) of pp. 105 [Mil04].

A **model** $M^o(G, X)$ of $Sh(G, X)$ over the reflex field $E(G, X)$ is called **canonical** if

- 1) $M^o(G, X)$ is equipped with a right action of $G(\mathbb{A}_f)$ that induces an equivariant isomorphism $M^o(G, X)_{\mathbb{C}} \cong Sh(G, X)$, and
- 2) for every special pair $(T, x) \subset (G, X)$ and $g \in G(\mathbb{A}_f)$ the point $[x, g] \in M^o(G, X)$ is rational over $E(x)^{ab}$ and the action of $\sigma \in Gal(E(x)^{ab}/E(x))$ is given by

$$\sigma[x, g] = [x, r_x(\nu)g] \quad (142)$$

where $\nu \in \mathbb{A}_{E(x), f}^{\times}$ is such that $[\nu] = \sigma^{-1}$ under Artin's reciprocity map.

In particular, for every compact open subgroup $K \subset G(\mathbb{A}_f)$, it follows that $M_K^o(G, X) = M^o(G, X)/K$ is a model of $Sh_K(G, X)$ over $E(G, X)$.

Remark 33. Canonical models are known to exist for all Shimura varieties (see [Mil04]).

D.7 Canonical model of connected Shimura varieties

We refer to 2.7.10 [Del79] for the precise definition of the canonical model $M^o(G, X^+)$ of a connected Shimura variety $Sh(G, X^+)$. Here we just want to mention the compatibility

$$M^o(G^{der}, X^+) = M^o(G, X)^o \quad (143)$$

where the latter denotes a correctly chosen connected component of the canonical model $M^o(G, X)$.

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